Optimal Risk Taking with Flexible Income*

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Abstract

We study the portfolio selection problem of an investor who can optimally exert costly effort for more income. The possibility of generating more income, if necessary, increases the risk-taking appetite of the investor. We find the optimal allocation to the risky security as a proportion of financial wealth and as a proportion of the total wealth, defined as the combination of the financial wealth and the human capital of the investor. When the investor's objective is the maximization of the terminal wealth, we show that the optimal allocation to the risky security is a hump-shaped function of the investment horizon. However, when the investor maximizes utility from intertemporal consumption, the optimal allocation in the risky security is a constant proportion of the total wealth of the investor.

Key Words: Utility Maximization, Optimal Portfolio Selection, Intertemporal Consumption, Optimal Effort.
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1 Introduction

The financial literature on optimal portfolio allocation starts with Samuelson (1969) and Merton (1969, 1971). The models of these influential papers, however, are based on a constant opportunity set and a given initial amount of wealth, but not intertemporal endowment or labor income. For the case of CRRA utility, almost standard in intertemporal problems, the optimal investment allocation is a constant proportion of wealth in the risky security.

Over the years, the literature on optimal portfolio allocation has expanded in a number of different directions. Of particular interest is the analysis of optimal portfolio allocation in the presence of labor income. Most of the work in this literature focuses on the effect of labor income on the optimal portfolio allocation over the working lifetime of the investor. On one hand, the empirical literature tries to find evidence about the age effect of portfolio allocation. Up to recently, most of the literature used cross-sectional data and found a negative relationship between age and proportion of wealth invested in stocks (for example, Bodie and Crane, 1997). More recently, Ameriks and Zeldes (2004), using panel data, do not find evidence of an age effect on investors allocation; however, as a good part of the empirical literature, they also document the low holdings of stock by young investors. On the other hand, most of the theoretical models study optimal allocation in the presence of labor income risk that cannot be totally hedged, like Svensson and Werner (1993), who solve the problem explicitly for exponential utility. In another model with exponential utility and non-hedgeable labor income, Henderson (2005) finds that portfolio weights are not monotone in time. Koo (1998) shows that when income risk is non-hedgeable, optimal consumption and allocation in the risky security are lower than what they would be in the complete markets case. Viceira (2001) shows that if stock returns and labor income are not correlated, an employed investor will invest more in stocks than a retired investor. A strand of this literature (for example, Heaton and Lucas (1997) and Campbell, Cocco, Gomes and Maenhout (2001))argues that low holdings of stock by young investors can be optimal only if labor income and stock returns are highly correlated (which is not the case). Another related paper is Cocco, Gomes and Maenhout (2005), which shows that labor income is an important factor in determining optimal portfolio allocation, but labor income risk does not have a large negative effect on utility. An alternative explanation to low holdings of stock by young investors is provided by Benzoni, Collin-Dufresne and Goldstein (2006), which studies the portfolio allocation when wages and stock returns are cointegrated (as opposed to perfectly correlated).

In this paper we are interested in a closely related but different setting, in which the investor optimally selects the rate of labor income, as opposed to the exogenous labor income process in the above mentioned theoretical work. There is also an old literature on endogenous labor/savings decisions, when labor supply is flexible, but most of that literature focuses on the choice of the optimal level of labor as affected by wage uncertainty. A recent paper by Flodén (2006) has an excellent survey of that literature. He finds that higher wage uncertainty implies higher labor effort. However, we are interested in the effect of stock returns uncertainty on the labor decision and, especially, on the investment allocation decision, assuming that the wages are constant.

The closest paper to the work we present here is Bodie, Merton and Samuelson (1992). There are two main differences between our model and theirs, motivated by both tractability and interpretation reasons. In their paper, they model the trade-off between consumption and leisure (labor is the difference between total available time and leisure) as a Cobb-Douglas function. Additionally, the analysis of results is limited to the case of logarithmic (myopic) preferences. In this paper, we model preferences over wealth and consumption as general (non-myopic) CRRA utility. Additionally, utility from wealth/consumption and disutility from effort are separable, with the disutility of effort modeled as a quadratic function. Effort in our model seems more suitable to be interpreted as extra labor to be supplied on an "exceptional" basis, with a cost that increases rapidly, and is independent of the level of comfort of the investor. Other relevant work is Basak (1999), who provides comparative static analysis of the effects of the labor-leisure choice on equilibrium consumption and stock prices, but does not provide an explicit portfolio allocation.

We extend and modify the findings of Bodie, Merton and Samuelson (1992). Consistent with their findings, we show that the possibility to work harder in the future, if necessary, affects the optimal risk-taking strategy of the investor. However, we find that the optimal strategy depends heavily on whether the investor cares about intertemporal consumption or only about final wealth (Wachter (2002) raises this issue in a setting without labor income). If the investor maximizes utility from intertemporal consumption, optimal investment allocation (like in Samuelson (1969) and Merton (1969), (1971)) is a constant proportion of wealth, if we consider total wealth, defined as the sum of financial wealth and human capital wealth (that is, the value of the future labor income). When the investor only cares about utility of final wealth, we show that the optimal portfolio allocation is a non-monotonic function of time. In particular, optimal allocation starts at a low level (as documented in the empirical literature we mentioned before) and increases over time, as the investor tries to substitute financial wealth for human capital wealth, up to a maximum level and then starts decreasing again for short horizons, since the possibility of future labor income shrinks.

The reminder of the paper is organized as follows: section 2 introduces the model and sets up the worker/investor problem. In section 3 we derive the solution to the problem. In section 4 we present several numerical exercises and present the main economic implications. We conclude in section 5.

2 The Problem

We consider the problem of an individual who faces a horizon of T years. We denote the wealth of this individual at time t by $X(t)$, with $X(0) = x$. This investor can allocate wealth between a risky asset and a risk-free security. The price of the risky asset follows a Geometric Brownian Motion Process:

$$dS(t) = \xi S(t)dt + \sigma S(t)dW(t),$$

with ξ and σ constant. The riskfree security pays a constant interest rate r, continuously compounded. The investor can either invest or borrow at this rate.

In addition, this individual has the option to exert compensated but costly effort, which can be interpreted as labor income. We assume that the wage is fixed and constant (we could allow for a time-dependent and deterministic wage, however). We denote the effort process by $u(t)$. We assume that

$$K_1 \leq u(t) \leq K_2 \quad .$$

Here, K_1 and K_2 represent some physical limits to the amount of extra effort the investor can exert. In general, we will assume $K_1 = 0$. As we will see, K_2 is an important parameter of the model because it restricts the income the investor can earn.[1]

Denote by π the dollar amount invested by this individual in the risky security; denote by $\theta = \frac{\xi - r}{\sigma}$ the market price of risk; finally, $c(t)$ represents the rate of consumption of this investor at time t. The wealth process of this investor satisfies the following dynamics:

$$dX(t) = \delta u(t)dt - c(t)dt + rX(t)dt + \pi(t)\sigma\theta dt + \pi(t)\sigma dW(t) \quad .$$

where δ is a positive constant that represents the skill or human capital of the individual and, therefore, has a direct effect on the future income prospects of the individual.

This investor has to choose the optimal processes $\pi(t)$, $u(t)$ that maximize expected CRRA utility of terminal wealth and consumption in excess of the labor cost,

$$E\left[\lambda \frac{X(T)^\gamma}{\gamma} + \mu \int_0^T e^{-sr^c} \frac{c(s)^\alpha}{\alpha} ds - \frac{1}{2} \int_0^T e^{-sr^u} u^2(s) ds)\right] \quad , \tag{1}$$

where λ, μ, r^c and r^u are constants. In particular, λ and μ represent the relative weights of the maximization of utility from terminal wealth versus the utility from intertemporal consumption.[2] The parameters r^c and r^u represent subjective discount

[1] We can think of $u(t)$ as a proxy for the labor time, in which case K_2 would be a measure of the employment limitation: a proxy for the the maximum number of hours the investor can work.

[2] It is standard to assume that $\lambda = \mu = 1$. However, as we show later, this does not seem a reasonable in our model with labor income. Obviously, we could set one of them equal to 1 and only change the other, but keeping both is more convenient for our numerical computations.

factors. Note that we do not assume a similar discount factor for final wealth since it will be implicit in the choice of λ.

We introduce the following notation:[3]

$$\tilde{Z}_t = e^{-rt-t\theta^2/2-\theta W_t}$$

$$\bar{T}(t) \;=\; \begin{cases} e^{tr^u}\frac{e^{(\theta^2-2r+r^u)(T-t)}-1}{\theta^2-2r+r^u} & \text{if } \theta^2 \neq 2r-r^u \\ e^{tr^u}e^{-\theta^2 t}(T-t) & \text{if } \theta^2 = 2r-r^u \end{cases}. \tag{2}$$

$$p(\alpha) = \frac{\alpha}{\alpha-1},$$

$$\rho_\alpha = p(\alpha)(p(\alpha)-1)\theta^2/2 - p(\alpha)r + \frac{1}{\alpha-1}r^c \tag{3}$$

$$\bar{Q}_\alpha(t) \;=\; \begin{cases} e^{\frac{tr^c}{\alpha-1}}\frac{e^{(\rho_\alpha)(T-t)}-1}{\rho_\alpha} & \text{if } \rho_\alpha \neq 0 \\ e^{\frac{tr^c}{\alpha-1}}e^{-\rho_\alpha t}(T-t) & \text{if } \rho_\alpha = 0 \end{cases}. \tag{4}$$

The following result shows the explicit solution of the previous maximization problem (1) when there is no upper limit to the amount of effort the investor can exert.

Theorem 1. *Consider the maximization problem (1). Assume $K_1 = 0$ and $K_2 = \infty$. The optimal amount of wealth $\hat{\pi}(t)$ to be invested in the risky security and optimal effort $\hat{u}(t)$ are given by the following expressions:*

$$\hat{\pi}_t = \frac{\theta}{\sigma}\left[\frac{1}{1-\gamma}(\frac{\hat{z}}{\lambda})^{\frac{1}{\gamma-1}}e^{\rho_\gamma(T-t)}(\tilde{Z}_t)^{\frac{1}{\gamma-1}} + \delta^2\hat{z}\tilde{Z}_t\bar{T}(t) + \frac{1}{1-\alpha}(\frac{\hat{z}}{\mu})^{\frac{1}{\alpha-1}}\bar{Q}_\alpha(t)(\tilde{Z}_t)^{\frac{1}{\alpha-1}}\right]. \tag{5}$$

and

$$\hat{u}(t) = \delta\hat{z}e^{tr^u}\tilde{Z}_t \tag{6}$$

The constant $\hat{z}$ is the solution of the following equation:

$$x = (\frac{\hat{z}}{\lambda})^{\frac{1}{\gamma-1}}e^{-\frac{\gamma}{\gamma-1}rT+\frac{\gamma}{2(\gamma-1)^2}\theta^2 T} - \hat{z}\delta^2\bar{T}_0 + (\frac{\hat{z}}{\mu})^{\frac{1}{\alpha-1}}\bar{Q}_\alpha(0). \tag{7}$$

The optimal consumption is given by

$$\hat{c}(t) = (\frac{\hat{z}}{\mu}e^{tr^c}\tilde{Z}_t)^{\frac{1}{\alpha-1}}.$$

Proof. See Appendix A.

[3] $\tilde{Z}_t$ is the state-price density process.

The next result is intuitive and compares the optimal portfolio allocation in this case with that of the standard "Merton" case, by breaking down the optimal allocation into two components, one of which is the "Merton" allocation.

Proposition 1. *Assume either utility from consumption only (that is, $\lambda = 0$), or from terminal wealth only (that is, $\mu = 0$), or $\alpha = \gamma$. Denote by w_t^M and $\hat{w}_t$ the optimal proportions of wealth (portfolio weights) to be invested in the risky security in the standard "Merton" problem and in our problem, respectively. Then $\hat{w}_t > w_t^M$. That is, the individual unambiguously takes on more risk in the presence of the option to be able to exert costly effort for more wealth in the future.*

Proof. See Appendix A.

This result shows that the option to work more in the future gives rise to more investment in the risky security. That is, the option to vary the labor supply in the future induces the individual to take on more risk today.

Now we consider the more realistic case in which there is an upper bound on the amount of effort, that is $K_2 < \infty$. This is a proxy for physical limitations or constraints on overtime or multiple jobs.

Theorem 2. *Consider the maximization problem (1). Assume $K_2 < \infty$. Then the optimal effort is given by*

$$\hat{u}(t) = \delta \hat{z} e^{tr^u} \tilde{Z}_t \mathbf{1}_{\{K_1 \leq \delta \hat{z} e^{tr^u} \tilde{Z}_t \leq K_2\}} + K_1 \mathbf{1}_{\{K_1 > \delta \hat{z} e^{tr^u} \tilde{Z}_t\}} + K_2 \mathbf{1}_{\{\delta \hat{z} e^{tr^u} \tilde{Z}_t > K_2\}} \tag{8}$$

The optimal consumption is of the same form as in Theorem 1. If we assume in addition $K_1 = 0$, the constant $\hat{z}$ is obtained from equation (19) *in the Appendix A. The optimal amount of wealth $\hat{\pi}(t)$ to be invested in the risky security is given by equation* (20) *in the Appendix A. Moreover, if we assume either utility from consumption only, or from terminal wealth only, or $\alpha = \gamma$, then the optimal portfolio is given by $\hat{\pi}(t) = \pi(t)^M + \pi(t)^L$, where*

$$\pi(t)^M = \frac{\theta}{\sigma(1-\gamma)} \hat{X}(t), \qquad \pi(t)^L = F(\gamma, r, T, \theta, \delta, K_2, \hat{z}\tilde{Z}_t)$$

where the functional form of F is given explicitly in Appendix A.

Proof. See Appendix A.

In the next section we present numerical exercises of the optimal portfolio allocation, using different parameter values, and discuss them.

For interpretational purposes, however, it will be useful to consider the optimal allocation, not only as a proportion of X, the "financial wealth" of the investor, but as a proportion of "total wealth" as well. The investor, we consider in our model, has two sources of wealth. First, his/her financial wealth, measured by X. In addition, this individual has human capital that, through costly effort, generates additional income. The literature considers this more general notion of wealth for the analysis of the optimal asset allocation. We consider both, the financial and the total wealth for the analysis of the optimal allocation.

Next we compute the human capital component of the investor's wealth. Since we have a complete markets setting, the obvious way to measure the human capital is by considering the future optimal (state-contingent) income as the payoff of a contingent claim. The price of the contingent claim in a complete markets setting is the present value of the expected payoff under the risk-neutral probability measure. We present that result in the following proposition.

Proposition 2. *The present value of the optimal future income under the risk-neutral probability measure is*

$$LW := H(\gamma, r, T, \theta, \delta, K_2, \hat{z}) \tag{9}$$

where the functional form of H is computed explicitly in the Appendix B.

Proof. See Appendix B.

Now we have explicit forms of the financial wealth and human capital of the investor, and the optimal allocation to the risky security. These enable us to compute and study the optimal risk-taking as a proportion of the financial or the "total wealth". We report the results in the next section.

3 Analysis of Results and Empirical implications

In this section we present numerical results for the model discussed in the previous section, according to the formulas in Theorem 2 and Proposition 2. The formulas are semi-analytical, but we can compute the results numerically. To simplify our computations, we assume that the initial financial wealth of the investor is one unit, that is, $X(0) = 1$. The optimal dollar investment allocation in the risky security is computed as explained in Theorem 1. We break this amount in two components, π^L and π^M. The term π^M is the optimal dollar amount that an investor in the Merton (1971) setting, with constant coefficients and no labor income, would allocate in the risky security; this is equal to

$$\pi^M(0) = \frac{\theta}{(1-\alpha)\sigma} X(0).$$

π^L is then $\pi - \pi^M$, and represents the part of the allocation in the risky security due to the future discretionary labor income of the investor. Obviously, π^L is always positive, since the investor can generate extra income, over the Merton (1971) allocation that we use as benchmark.

In addition, we report the proportion of "total wealth," not just financial wealth, invested in the risky security. Total wealth is the sum of financial wealth X and human capital wealth, LW from Proposition 2.

Our results are presented in tables 1-4. One problem we face is the choice of reasonable parameter values. That is straightforward for such parameters as the degree of risk-aversion, the market price or risk or interest rates, which are directly observable or have been extensively discussed in the literature. However, we do not have a good benchmark for some other parameters in our model, namely δ, which represents the "quality" of the human capital of the worker, and K_2, which represents the upper limit on the amount of effort (say due to time availability) that the worker/investor can exert. We have chosen the values for δ, K_2 , μ and λ so that the following conditions are satisfied: First, the expected annual income from labor is similar to the initial financial wealth x of the investor; second, the expected annual consumption is about half of the expected annual income; third the expected annual increase in wealth is of the order of 10 %; and fourth, the worker's wealth increases but may not more than double in a year.[4] Since our objective is to study the sensitivity of the optimal allocation to different parameters of the model, we perturb the parameters and study the impact of each on the optimal allocation.

Here are the main conclusions of our comparative statics analysis:

(i) Our main finding is that, in the presence of the flexible labor income, the optimal investment strategy of an investor who maximizes utility of terminal wealth is qualitatively different from the one of the investor whose objective is to maximize utility of intertemporal consumption.

(ii) In particular, when the investor only maximizes utility from terminal wealth, the proportion of wealth (either just financial wealth or total wealth, including human capital) changes with respect to the time horizon in a non-monotonic way; for reasonable parameter values, the proportion of holdings in the stock initially increases, but then decreases as the time horizon decreases. From our comparative statics analysis, this relationship between the investment horizon and optimal allocation seems robust, and not the result of changes in the human capital wealth.

(iii) When the investor cares about intertemporal consumption exclusively, the optimal investment strategy requires to hold a constant proportion of the total wealth in the stock (which yields a decreasing proportion of financial wealth) as the investor approaches the end of the horizon. This proportion is identical to that in the Merton's benchmark.

(iv) In fact, in order for the utility from final wealth to matter, we have to allocate a very small weight (close to zero) to the utility from intertemporal consumption.

(v) When the investor only cares about utility of final wealth, π^L, the investment in the risky security due to the possibility of discretionary labor income, decreases with risk-aversion, but the rate at which it does depends on the horizon. For a very long horizon, the proportional decrease is larger than in the Merton's benchmark case. For short horizons, the proportional decrease is smaller. As

[4] We have chosen K_2 so that it would be binding (that is, the worker/investor would optimally generate more labor income for higher K_2); in addition, it is such that the value of LW would not be of a different order of magnitude than in the case of non-binding K_2.)

we have explained before, when the investor cares about intertemporal consumption, the effect of risk aversion on the proportion of total wealth invested in the risky security is the same as in Merton's benchmark case.

Some of the results are completely new and striking. In particular, the effect of the horizon in (i)-(iii). Intertemporal consumption has the effect of effectively "shortening" the horizon of the investor and the proportion of wealth allocated in the risky security is constant (since relative risk-aversion is constant). This is consistent with the intuition provided in Wachter (2002), which compares the effect of intertemporal consumption with the idea of the duration of a bond as a measure of actual maturity. When the investor only cares about utility of final wealth, the horizon affects the proportion invested in the risky stock. This effect is non-monotonic as a result of two opposing forces. First, as the investor horizon becomes shorter, the investor allocates a larger proportion of wealth into the risky asset in order to accumulate wealth faster. The possibility of getting additional labor income later makes the investor less risk-averse. However, as the horizon shortens, the human capital wealth decreases, which makes the investor more risk-averse and, at some point, leads to a reduction in the wealth allocated into the risky security. Result (iv), is due to the different "measure" (a point versus a continuum) of the intertemporal utility (from consumption) versus the one-period utility (from final wealth). The intuition of (v) is similar to that of (ii)-(iii): the difference in optimal strategies between two investors with different degrees of risk-aversion is larger than in the Merton's benchmark case since the long-horizon makes both investors feel less risk-averse and scales up the difference in risk-aversion. When the horizon is short, the value of flexible labor income is small and the part of the optimal allocation due to it converges to zero for both investors.

Overall, it is clear that the possibility of future extra labor income can affect investment decisions in a substantial way and cannot be ignored in optimal portfolio allocation analysis, or in equilibrium considerations, like the equity risk premium puzzle of Mehra and Prescott (1985): as we have shown, a long-time horizon of the objective of the investor makes the investor effectively less risk-averse, which actually would make the puzzle even stronger, and as a result, an even higher degree of risk aversion would be needed to explain it.

4 Conclusion

We consider a model in which the worker/investor chooses the level of labor income by exerting costly effort. In our model there is no wage uncertainty (we assume the real wage is constant and equal to one). The possibility of flexible income has an important effect on the optimal portfolio allocation of the investor if the investor only cares about utility from final wealth. In that case, the optimal allocation to the risky security is age-dependent and is hump-shaped as a function of time.

However, when the investor maximizes the utility of intertemporal consumption, optimal allocation to the risky security is a constant proportion of wealth, when wealth includes both financial wealth and the value of future optimal labor income.

Thus, depending on the investor's objective, the dependence of the optimal allocation to risky security on the investor's age could be monotonic or hump-shaped.

Our model can shed light on some of the empirical puzzles documented in the literature on portfolio allocation. The solutions obtained in our paper are semi-analytic, which allows us to perform comparative static analysis of the relationship between various parameters and the optimal portfolio allocation. Our model offers a good platform for calibration. That exercise, however, is not trivial, and is left for future research. In our model we assume that the worker/investor has costly but sure access to labor income (up to the limit K_2).

Other interesting, but non-trivial, extensions to our model are the case of wage uncertainty, which has been widely studied in the literature, but only when the labor income is exogenous, and the case of mortality risk of the individual (or uncertain time horizon). These are left for future research.

5 Appendix A

Proof of Theorem 1.

Using the standard duality/martingale techniques, we immediately derive the expression for the optimal effort (6) and the following expressions for optimal terminal wealth and consumption:[5]

$$\hat{X}(T) = (\frac{\hat{z}}{\lambda}\tilde{Z}_T)^{\frac{1}{\gamma-1}}$$

$$\hat{c}(t) = (\frac{\hat{z}}{\mu}e^{tr^c}\tilde{Z}_t)^{\frac{1}{\alpha-1}}$$

The wealth process at time $t < T$ is obtained from

$$\tilde{Z}_t\hat{X}_t = E_t[\tilde{Z}_T(\frac{\hat{z}}{\lambda}\tilde{Z}_T)^{\frac{1}{\gamma-1}} - \delta\int_t^T \tilde{Z}(s)\hat{u}_s ds + \int_t^T \tilde{Z}_s(\frac{\hat{z}}{\mu}e^{tr^c}\tilde{Z}_s)^{\frac{1}{\alpha-1}}ds]. \tag{10}$$

For any given p we have the following property:

$$E_t[Z^p(T)] = E_t\left[e^{-p^2\theta^2T/2-p\theta W(T)}e^{p(p-1)\theta^2T/2}\right] = e^{-p^2\theta^2t/2-p\theta W(t)}e^{p(p-1)\theta^2T/2} \quad , \tag{11}$$

Using the property (11) and simplifying (10) we get,

$$\tilde{Z}_t\hat{X}_t = (\frac{\hat{z}}{\lambda})^{\frac{1}{\gamma-1}}e^{\rho_\gamma(T-t)}(\tilde{Z}_t)^{\frac{\gamma}{\gamma-1}} - \delta^2\hat{z}\tilde{Z}_t^2\bar{T}(t) + (\frac{\hat{z}}{\mu})^{\frac{1}{\alpha-1}}(\tilde{Z}_t)^{\frac{\alpha}{\alpha-1}}\bar{Q}_\alpha(t) \quad . \tag{12}$$

[5]The proof for these expressions is very similar to the one of Theorem 1 of Cadenillas, Cvitanić and Zapatero (2005).

The constant $\hat{z}$ is obtained by setting $t = 0$ above, that is,

$$x = (\frac{\hat{z}}{\lambda})^{\frac{1}{\gamma-1}} e^{-\frac{\gamma}{\gamma-1}rT+\frac{\gamma}{2(\gamma-1)^2}\theta^2 T} - \hat{z}\delta^2 \bar{T}_0 + (\frac{\hat{z}}{\mu})^{\frac{1}{\alpha-1}} \bar{Q}_\alpha(0) \quad . \tag{13}$$

Applying the Ito's formula, we get

$$dX = (\cdots)dt + \left[\frac{1}{1-\gamma}\theta(\frac{\hat{z}}{\lambda})^{\frac{1}{\gamma-1}} e^{\rho_\gamma(T-t)}(\tilde{Z}_t)^{\frac{1}{\gamma-1}} + \theta\delta^2\hat{z}\tilde{Z}_t\bar{T}(t) + \frac{1}{1-\alpha}\theta(\frac{\hat{z}}{\mu})^{\frac{1}{\alpha-1}}\bar{Q}_\alpha(t)(\tilde{Z}_t)^{\frac{1}{\alpha-1}}\right] dW \quad .$$

The expression in the [ ] brackets has to be equal to $\sigma\pi$, which gives us the optimal amount in stock $\hat{\pi}$ as

$$\hat{\pi}_t = \frac{\theta}{\sigma}\left[\frac{1}{1-\gamma}(\frac{\hat{z}}{\lambda})^{\frac{1}{\gamma-1}} e^{\rho_\gamma(T-t)}(\tilde{Z}_t)^{\frac{1}{\gamma-1}} + \delta^2\hat{z}\tilde{Z}_t\bar{T}(t) + \frac{1}{1-\alpha}(\frac{\hat{z}}{\mu})^{\frac{1}{\alpha-1}}\bar{Q}_\alpha(t)(\tilde{Z}_t)^{\frac{1}{\alpha-1}}\right] \quad . \tag{14}$$

□

Proof of Proposition 1. Under the assumptions of the proposition, we can rewrite (14) in the following form:

$$\hat{\pi}_t = \frac{\theta}{\sigma}\left[\frac{1}{1-\gamma}X_t + (\frac{1}{1-\gamma}+1)\delta^2\hat{z}\tilde{Z}_t\bar{T}(t)\right] \quad = \pi_t^M + \pi_t^L \quad , \tag{15}$$

where

$$\pi_t^L = \frac{\theta}{\sigma}\left[(\frac{1}{1-\gamma}+1)\delta^2\hat{z}\tilde{Z}_t\bar{T}(t)\right] \quad .$$

We need to show that π_t^L is positive. Since $\gamma < 1$ we just need to show that $\hat{z} > 0$. It follows from the equation (13) that the right hand side (RHS) of (13) converges to $+\infty$ as $\hat{z} \to 0$ (since $\frac{1}{1-\gamma} < 0$), and the RHS of (13) converges to $-\infty$, as $\hat{z} \to +\infty$. Since $x > 0$ and the RHS is a continuous function of $\hat{z}$, then there exists a $\hat{z} \in (0,\infty)$ that solves (13). That is there exists a positive solution $\hat{z}$ of (13) , which implies that $\pi_t^L > 0$ and thus, $\hat{\pi}_t > \pi_t^M$, and also $\hat{w}_t > w_t^M$.

□

Proof of Theorem 2.

Expressions for the optimal terminal wealth and optimal effort are derived as above, in the proof of Theorem 1. For $K_1 = 0$ we have,

$$E_t[\delta\int_t^T \tilde{Z}(s)u_s ds] = \delta K_2\tilde{Z}_t e^{rt}\int_t^T e^{-rs}N(d_1(s-t,\hat{z}e^{tr^u}\tilde{Z}_t))ds +$$

$+$

$$\delta^2\hat{z}\tilde{Z}_t^2 e^{2(r-\theta^2/2)t}\int_t^T e^{-2(r-\theta^2/2)s}e^{sr^u}N(d_2(s-t,\hat{z}e^{tr^u}\tilde{Z}_t))ds \tag{16}$$

where

$$d_1(s-t,\hat{z}) := \frac{1}{\theta\sqrt{s-t}}[\log(\delta\hat{z}) - \log(K_2) - (r - r^u - \theta^2/2)(s-t)] \qquad (17)$$

and

$$d_2(s-t,\hat{z}) := \frac{1}{\theta\sqrt{s-t}}[-\log(\delta\hat{z}) + \log(K_2) + (r - r^u - 3\theta^2/2)(s-t)] \quad . \qquad (18)$$

and $N(.)$ is the cdf of the standard normal distribution. From the equation (16) we can derive the dynamics of the wealth process, assuming $K_1 = 0$:

$$\begin{aligned} \hat{X}_t &= e^{\rho_\gamma(T-t)}(\frac{\hat{z}}{\lambda}\tilde{Z}_t)^{\frac{1}{\gamma-1}} + \bar{Q}_\alpha(t)(\frac{\hat{z}}{\mu}\tilde{Z}_t)^{\frac{1}{\alpha-1}} \\ &- \delta K_2 \int_t^T e^{-r(s-t)} N(d_1(s-t,\hat{z}e^{tr^u}\tilde{Z}_t))ds \\ &- \delta^2\hat{z}\tilde{Z}_t \int_t^T e^{-2(r-\theta^2/2)(s-t)} e^{sr^u} N(d_2(s-t,\hat{z}e^{tr^u}\tilde{Z}_t))ds \quad . \end{aligned}$$

The constant $\hat{z}$ is obtained by setting $t = 0$, that is,

$$x = e^{\rho_\gamma T}(\frac{\hat{z}}{\lambda})^{\frac{1}{\gamma-1}} + \bar{Q}_\alpha(0)(\frac{\hat{z}}{\mu})^{\frac{1}{\alpha-1}} -$$

$$-\delta K_2 \int_0^T e^{-rs} N(d_1(s,\hat{z}))ds - \delta^2\hat{z}\int_0^T e^{-2(r-\theta^2/2)s} e^{sr^u} N(d_2(s,\hat{z}))ds \quad . \qquad (19)$$

Applying the Ito's lemma, we observe that the coefficient of the noise term dW_t in the dynamics of dX_t is

$$\begin{aligned} \sigma\pi_t &= \frac{1}{1-\gamma}\theta e^{\rho_\gamma(T-t)}(\frac{\hat{z}}{\lambda}\tilde{Z}_t)^{\frac{1}{\gamma-1}} + \frac{1}{1-\alpha}\theta\bar{Q}_\alpha(t)(\frac{\hat{z}}{\mu}\tilde{Z}_t)^{\frac{1}{\alpha-1}} \\ &+ \theta\delta^2\hat{z}\tilde{Z}_t \int_t^T e^{-2(r-\theta^2/2)(s-t)} e^{sr^u} N(d_2(s-t,\hat{z}e^{tr^u}\tilde{Z}_t))ds \\ &+ \delta K_2 \int_t^T \frac{e^{-r(s-t)}}{\sqrt{s-t}} n(d_1(s-t,\hat{z}e^{tr^u}\tilde{Z}_t))ds \\ &- \delta^2\hat{z}\tilde{Z}_t \int_t^T \frac{e^{-2(r-\theta^2/2)(s-t)}e^{sr^u}}{\sqrt{s-t}} n(d_2(s-t,\hat{z}e^{tr^u}\tilde{Z}_t))ds \quad . \end{aligned} \qquad (20)$$

If $\alpha = \gamma$, or we have utility from terminal wealth only, or we have utility of consumption only, then using (19) and (20) we can get the following form for π_t:

$$\begin{aligned}
\pi_t &= \frac{\theta}{\sigma(1-\gamma)}\hat{X}_t + \frac{\theta\delta K_2}{\sigma(1-\gamma)}\int_t^T e^{-r(s-t)}N(d_1(s-t,\hat{z}e^{tr^u}\tilde{Z}_t))ds \\
&+ \frac{\theta\delta^2\hat{z}\tilde{Z}_t}{\sigma}\left(\frac{1}{1-\gamma}+1\right)\int_t^T e^{-(2r-\theta^2)(s-t)}e^{sr^u}N(d_2(s-t,\hat{z}e^{tr^u}\tilde{Z}_t))ds \\
&+ \frac{\delta K_2}{\sigma}\int_t^T \frac{e^{-r(s-t)}}{\sqrt{s-t}}n(d_1(s-t,\hat{z}e^{tr^u}\tilde{Z}_t))ds \\
&- \frac{\delta^2\hat{z}\tilde{Z}_t}{\sigma}\int_t^T \frac{e^{-(2r-\theta^2)(s-t)}e^{sr^u}}{\sqrt{s-t}}n(d_2(s-t,\hat{z}e^{tr^u}\tilde{Z}_t))ds \quad ,
\end{aligned} \tag{21}$$

where $n(x)$ is the standard normal density function. The equation (21) can be written in the form $\hat{\pi}_t = \pi_t^M + \pi_t^L$, where

$$\pi_t^L = F(\gamma, r, T, \theta, \delta, K_2, \hat{z}\tilde{Z}_t) = \frac{\theta\delta K_2}{\sigma(1-\gamma)}\int_t^T e^{-r(s-t)}N(d_1(s-t,\hat{z}e^{tr^u}\tilde{Z}_t))ds \; +$$

$$+\frac{\theta\delta^2\hat{z}\tilde{Z}_t}{\sigma}\left(\frac{1}{1-\gamma}+1\right)\int_t^T e^{-(2r-\theta^2)(s-t)}e^{sr^u}N(d_2(s-t,\hat{z}e^{tr^u}\tilde{Z}_t))ds$$

$$+\frac{\delta K_2}{\sigma}\int_t^T \frac{e^{-r(s-t)}}{\sqrt{s-t}}n(d_1(s-t,\hat{z}e^{tr^u}\tilde{Z}_t))ds - \frac{\delta^2\hat{z}\tilde{Z}_t}{\sigma}\int_t^T \frac{e^{-(2r-\theta^2)(s-t)}e^{sr^u}}{\sqrt{s-t}}n(d_2(s-t,\hat{z}e^{tr^u}\tilde{Z}_t))ds$$

This proves the theorem.[6]

□

Remark 4: Taking $t = 0$ in (21), we have the following expression:

$$\pi_0^L = \frac{\theta\delta K_2}{\sigma(1-\gamma)}\int_0^T e^{-rs}N(d_1(s,\hat{z}))ds \; +$$

$$+\frac{\theta\delta^2\hat{z}}{\sigma}\left(\frac{1}{1-\gamma}+1\right)\int_0^T e^{-(2r-\theta^2)s}e^{sr^u}N(d_2(s,\hat{z}))ds$$

$$+\frac{\delta K_2}{\sigma}\int_0^T \frac{e^{-rs}}{\sqrt{s}}n(d_1(s,\hat{z}))ds - \frac{\delta^2\hat{z}}{\sigma}\int_0^T \frac{e^{-(2r-\theta^2)s}e^{sr^u}}{\sqrt{s}}n(d_2(s,\hat{z}))ds \tag{22}$$

The details of the calculations of the above integrals are given in the Appendix B. We use certain properties, given in the next remark.

Remark 5: The calculations of the integrals in (21) are done, by using the following properties and formulas. Denote

[6] See the remarks and the properties of this Appendix and of Appendix B for simplifications of the results and the derivations of the closed-form expressions for the optimal portfolio proportions.

$$N(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{t^2}{2}}dt,\qquad n(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

and

$$N(x,y)=N\left(\frac{\sqrt{T-t}}{\theta}y+\frac{x}{\theta\sqrt{T-t}}\right) \tag{23}$$

Then we have the following properties:

Property 1.

$$\begin{aligned}
&\int_t^T e^{-rs}N\left(\frac{\sqrt{s-t}}{\theta}y+\frac{\log(x)}{\theta\sqrt{s-t}}\right)ds\\
&=\mathbf{1}_{\{x>1\}}+\frac{1}{2}\mathbf{1}_{\{x=1\}}-e^{-r(T-t)}N\left(x,-r+\theta^2/2\right)\\
&+(x)^{\frac{\theta^2/2-3r}{\theta^2}}\left\{\frac{-r+\theta^2/2}{r+\theta^2/2}\left[N\left(x,r+\theta^2/2\right)-\mathbf{1}_{\{x>1\}}-\frac{1}{2}\mathbf{1}_{\{x=1\}}\right]\right.\\
&+\frac{1}{2}\left[1-\frac{-r+\theta^2/2}{r+\theta^2/2}\right]\left[N\left(x,r+\theta^2/2\right)+(x)^{\frac{-2r-\theta^2}{\theta^2}}N\left(x,-r-\theta^2/2\right)\right.\\
&\left.\left.-\left[1+(x)^{\frac{-2r-\theta^2}{\theta^2}}\right]\mathbf{1}_{\{x>1\}}-\mathbf{1}_{\{x=1\}}\right]\right\}
\end{aligned} \tag{24}$$

Property 2.

$$\begin{aligned}
&\int_t^T \frac{e^{-rs-t}}{\sqrt{s-t}}n\left(\frac{1}{\theta}(y\sqrt{s-t}+\frac{\log(x)}{\sqrt{s-t}})\right)ds\\
&=\frac{2\theta}{y}\left[N\left(\frac{1}{\theta}(y\sqrt{T-t}+\frac{\log(x)}{\sqrt{T-t}})\right)-\mathbf{1}_{\{x>1\}}-\frac{1}{2}\mathbf{1}_{\{x=1\}}\right]\\
&-\frac{2\theta}{2y}\left[N\left(\frac{1}{\theta}(y\sqrt{T-t}+\frac{\log(x)}{\sqrt{T-t}})\right)+(x)^{-\frac{2y}{\theta^2}}N\left(\frac{1}{\theta}(-y\sqrt{T-t}+\frac{\log(x)}{\sqrt{T-t}})\right)\right.\\
&\left.-\left[1+(x)^{-\frac{2y}{\theta^2}}\right]\mathbf{1}_{\{x>1\}}-\mathbf{1}_{\{x=1\}}\right]
\end{aligned} \tag{25}$$

Property 3.

$$\frac{an\left(\frac{a}{\sqrt(s)}+b\sqrt(s)\right)}{s^{3/2}}=-N'\left(\frac{a}{\sqrt(s)}+b\sqrt(s)\right)-e^{-2ab}N'\left(\frac{a}{\sqrt(s)}-b\sqrt(s)\right)$$

Property 4.

$$\frac{n\left(\frac{a}{\sqrt(s)}+b\sqrt(s)\right)}{\sqrt{s}}=\frac{2}{b}N'\left(\frac{a}{\sqrt(s)}+b\sqrt(s)\right)+\frac{a}{b}\frac{n\left(\frac{a}{\sqrt(s)}+b\sqrt(s)\right)}{s^{3/2}}$$

$\square$

6 Appendix B

Define the following:

$$I_1(r,a,b,T) = \int_0^T e^{-rs} N(\frac{a}{\sqrt{s}} + b\sqrt{s})ds \tag{26}$$

$$I_2(r,a,b,T) = a\int_0^T \frac{e^{-rs}}{s^{3/2}} n(\frac{a}{\sqrt{s}} + b\sqrt{s})ds \tag{27}$$

$$I_3(r,a,b,T) = \int_0^T \frac{e^{-rs}}{s^{1/2}} n(\frac{a}{\sqrt{s}} + b\sqrt{s})ds \tag{28}$$

$$A_1(r,a,b,T) = \frac{1}{r}\left[\frac{1}{2}\mathbf{1}_{\{a=0\}} + \mathbf{1}_{\{a>0\}} - e^{-rT}N(\frac{a}{\sqrt{T}} + b\sqrt{T})\right] \tag{29}$$

$$A_2(r,a,b,T) = \frac{2e^{a(\sqrt{b^2+2r}-b)}}{\sqrt{b^2+2r}}\left[-\frac{1}{2}\mathbf{1}_{\{a=0\}} - \mathbf{1}_{\{a>0\}} + N(\frac{a}{\sqrt{T}} + \sqrt{b^2+2r}\sqrt{T})\right] \tag{30}$$

$$A_3(r,a,b,T) = e^{-a(\sqrt{b^2+2r}+b)}\left[-\frac{1}{2}\mathbf{1}_{\{a=0\}} - \mathbf{1}_{\{a>0\}} + N(\frac{a}{\sqrt{T}} - \sqrt{b^2+2r}\sqrt{T})\right] \tag{31}$$

Then using properties 1-4 and some algebra we get the following simplified formulas:

$$I_2(r,a,b,T) = -\frac{\sqrt{b^2+2r}}{2}A_2 - A_3 \tag{32}$$

$$I_3(r,a,b,T) = A_2(r,a,b,T) + \frac{1}{\sqrt{b^2+2r}}I_2(r,a,b,T) \tag{33}$$

$$I_1(r,a,b,T) = A_1(r,a,b,T) - \frac{1}{2r}I_2(r,a,b,T) + \frac{b}{2r}I_3(r,a,b,T) \tag{34}$$

To find $\hat{z}$ we solve the following equation numerically:

$$G(x,\gamma,r,T,\theta,\delta,K_2,\hat{z}) =$$

$$\delta K_2 I_1\left(r, \frac{ln(\frac{\delta\hat{z}}{K_2})}{\theta}, \frac{\theta^2/2 - r + r_u}{\theta}, T\right) + \delta^2\hat{z} I_1\left(2r - r_u - \theta^2, \frac{ln(\frac{K_2}{\delta\hat{z}})}{\theta}, \frac{r - r_u - 3\theta^2/2}{\theta}, T\right)$$

$$-(\frac{\hat{z}}{\lambda})^{\frac{1}{\gamma-1}}e^{-\frac{\gamma}{\gamma-1}rT+\frac{\gamma}{2(\gamma-1)^2}\theta^2T+\frac{r_cT}{\alpha-1}}-(\frac{\hat{z}}{\mu})^{\frac{1}{\alpha-1}}\bar{Q}_\alpha(0)+x=0 \quad . \tag{35}$$

Then the π_0^L can be found as

$$\pi_0^L=\frac{\theta\delta K_2}{\sigma(1-\gamma)}I_1\left(r,\frac{ln(\frac{\delta\hat{z}}{K_2})}{\theta},\frac{\theta^2/2-r+r_u}{\theta},T\right)+\frac{\delta K_2}{\sigma}I_3\left(r,\frac{ln(\frac{\delta\hat{z}}{K_2})}{\theta},\frac{\theta^2/2-r+r_u}{\theta},T\right)$$
$$+\frac{2-\gamma}{1-\gamma}\frac{\delta^2\hat{z}\theta}{\sigma}I_1\left(2r-r_u-\theta^2,\frac{ln(\frac{K_2}{\delta\hat{z}})}{\theta},\frac{r-r_u-3\theta^2/2}{\theta},T\right)$$
$$-\frac{\delta^2\hat{z}}{\sigma}I_3\left(2r-r_u-\theta^2,\frac{ln(\frac{K_2}{\delta\hat{z}})}{\theta},\frac{r-r_u-3\theta^2/2}{\theta},T\right) \quad . \tag{36}$$

Proof of Proposition 2.

The left side of the equation (16) is exactly the present value of the future labor income, under the martingale measure. Using that equation, along with Properties 1-4 and definitions of Appendix B, we get that

$$LW := H(\gamma,r,T,\theta,\delta,K_2,\hat{z}) =$$
$$\delta K_2 I_1\left(r,\frac{ln(\frac{\delta\hat{z}}{K_2})}{\theta},\frac{\theta^2/2-r+r_u}{\theta},T\right)+\delta^2\hat{z}I_1\left(2r-r_u-\theta^2,\frac{ln(\frac{K_2}{\delta\hat{z}})}{\theta},\frac{r-r_u-3\theta^2/2}{\theta},T\right) \tag{37}$$

□

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Table 1
Optimal allocation for different ability levels

In the table, T is the time horizon of the investor. LW is the value of the human capital wealth of the investor. π^L is the dollar amount invested in the risky security in excess of the optimal allocation in Merton's benchmark. π^M is the optimal allocation in Merton's benchmark. $\pi = \pi^L + \pi^M$ is the total allocation in the risky security. X is financial wealth, so that $\pi/(X + LW)$ is the proportion of total wealth (financial plus human capital) invested in the risky security. δ is the parameter that measures the ability of the investor. μ is the weight of intertemporal consumption in the utility function of the investor. Other parameter values of the model in this table are as follows: $\gamma = -5$, so that risk aversion $1 - \gamma = 6$; the upper bound in the effort is $K_2 = 0.2$; the weight of the utility of final wealth in total utility is $\lambda = 0.7$; interest rate is $r = 0.035$; market price of risk is $\theta = 0.3$; the discount factor for the cost of effort is $r_u = 0.05$; the subjective discount factor of the utility of intertemporal consumption is $r_c = 0.05$; finally, initial liquid wealth is $X = 1$.

$\delta = 0.3$

$\mu = 0.1$					$\mu = 0$				
T	LW	π^L	π^M	$\frac{\pi}{X+LW}$	T	LW	π^L	π^M	$\frac{\pi}{X+LW}$
1	0.059	0.0147	0.250	0.250	1	0.040	0.0662	0.250	0.304
5	0.275	0.0688	0.250	0.250	5	0.068	0.1175	0.250	0.344
10	0.506	0.1266	0.250	0.250	10	0.049	0.0854	0.250	0.320
15	0.700	0.1750	0.250	0.250	15	0.028	0.0479	0.250	0.290
20	0.863	0.2157	0.250	0.250	20	0.013	0.0231	0.250	0.273
25	1.000	0.2499	0.250	0.250	25	0.006	0.0102	0.250	0.259

$\delta = 0.6$

$\mu = 0.1$					$\mu = 0$				
T	LW	π^L	π^M	$\frac{\pi}{X+LW}$	T	LW	π^L	π^M	$\frac{\pi}{X+LW}$
1	0.118	0.0295	0.250	0.250	1	0.104	0.1288	0.250	0.343
5	0.550	0.1376	0.250	0.250	5	0.163	0.2781	0.250	0.454
10	1.013	0.2531	0.250	0.250	10	0.128	0.2193	0.250	0.416
15	1.400	0.3501	0.250	0.250	15	0.081	0.1397	0.250	0.360
20	1.726	0.4315	0.250	0.250	20	0.044	0.0764	0.250	0.313
25	1.999	0.4998	0.250	0.250	25	0.021	0.0368	0.250	0.281

Table 2
Optimal allocation for different upper bounds to the effort level

In the table, T is the time horizon of the investor. LW is the value of the human capital wealth of the investor. π^L is the dollar amount invested in the risky security in excess of the optimal allocation in Merton's benchmark. π^M is the optimal allocation in Merton's benchmark. $\pi = \pi^L + \pi^M$ is the total allocation in the risky security. X is financial wealth, so that $\pi/(X + LW)$ is the proportion of total wealth (financial plus human capital) invested in the risky security. δ is the parameter that measures the ability of the investor. μ is the weight of intertemporal consumption in the utility function of the investor. Other parameter values of the model in this table are as follows: $\gamma = -5$, so that risk aversion $1 - \gamma = 6$; the ability of the investor is $\delta = 0.3$; the weight of the utility of final wealth in total utility is $\lambda = 0.7$; interest rate is $r = 0.035$; market price of risk is $\theta = 0.3$; the discount factor for the cost of effort is $r_u = 0.05$; the subjective discount factor of the utility of intertemporal consumption is $r_c = 0.05$; finally, initial liquid wealth is $X = 1$.

$K_2 = 0.2$

$\mu = 0.1$					$\mu = 0$				
T	LW	π^L	π^M	$\frac{\pi}{X+LW}$	T	LW	π^L	π^M	$\frac{\pi}{X+LW}$
1	0.059	0.0147	0.250	0.250	1	0.040	0.0662	0.250	0.304
5	0.275	0.0688	0.250	0.250	5	0.068	0.1175	0.250	0.344
10	0.506	0.1266	0.250	0.250	10	0.049	0.0854	0.250	0.320
15	0.700	0.1750	0.250	0.250	15	0.028	0.0479	0.250	0.290
20	0.863	0.2157	0.250	0.250	20	0.013	0.0231	0.250	0.273
25	1.000	0.2499	0.250	0.250	25	0.006	0.0102	0.250	0.259

$K_2 = 0.8$

$\mu = 0.1$					$\mu = 0$				
T	LW	π^L	π^M	$\frac{\pi}{X+LW}$	T	LW	π^L	π^M	$\frac{\pi}{X+LW}$
1	0.236	0.0638	0.250	0.250	1	0.040	0.0695	0.250	0.307
5	1.101	0.2752	0.250	0.250	5	0.068	0.1195	0.250	0.346
10	2.025	0.5063	0.250	0.250	10	0.050	0.0866	0.250	0.337
15	2.801	0.7002	0.250	0.250	15	0.028	0.0484	0.250	0.298
20	3.452	0.8631	0.250	0.250	20	0.013	0.0233	0.250	0.270
25	3.999	0.9999	0.250	0.250	25	0.006	0.0102	0.250	0.260

Table 3
Optimal allocation for different degrees of risk aversion $(1-\gamma)$

In the table, T is the time horizon of the investor. LW is the value of the human capital wealth of the investor. π^L is the dollar amount invested in the risky security in excess of the optimal allocation in Merton's benchmark. π^M is the optimal allocation in Merton's benchmark. $\pi = \pi^L + \pi^M$ is the total allocation in the risky security. X is financial wealth, so that $\pi/(X + LW)$ is the proportion of total wealth (financial plus human capital) invested in the risky security. δ is the parameter that measures the ability of the investor. μ is the weight of intertemporal consumption in the utility function of the investor. Other parameter values of the model in this table are as follows: the ability of the investor is $\delta = 0.3$; the upper bound in the effort is $K_2 = 0.2$; the weight of the utility of final wealth in total utility is $\lambda = 0.7$; interest rate is $r = 0.035$; market price of risk is $\theta = 0.3$; the discount factor for the cost of effort is $r_u = 0.05$; the subjective discount factor of the utility of intertemporal consumption is $r_c = 0.05$; finally, initial liquid wealth is $X = 1$.

$\gamma = -5$

$\mu = 0.1$					$\mu = 0$				
T	LW	π^L	π^M	$\frac{\pi}{X+LW}$	T	LW	π^L	π^M	$\frac{\pi}{X+LW}$
1	0.059	0.0147	0.250	0.250	1	0.040	0.0662	0.250	0.304
5	0.275	0.0688	0.250	0.250	5	0.068	0.1175	0.250	0.344
10	0.506	0.1266	0.250	0.250	10	0.049	0.0854	0.250	0.320
15	0.700	0.1750	0.250	0.250	15	0.028	0.0479	0.250	0.290
20	0.863	0.2157	0.250	0.250	20	0.013	0.0231	0.250	0.273
25	1.000	0.2499	0.250	0.250	25	0.006	0.0102	0.250	0.259

$\gamma = -1$

$\mu = 0.1$					$\mu = 0$				
T	LW	π^L	π^M	$\frac{\pi}{X+LW}$	T	LW	π^L	π^M	$\frac{\pi}{X+LW}$
1	0.059	0.0451	0.750	0.750	1	0.051	0.0928	0.750	0.802
5	0.275	0.2070	0.750	0.750	5	0.155	0.3086	0.750	0.917
10	0.506	0.3807	0.750	0.750	10	0.197	0.3943	0.750	0.956
15	0.700	0.5267	0.750	0.750	15	0.201	0.4011	0.750	0.958
20	0.862	0.6494	0.750	0.750	20	0.189	0.3741	0.750	0.945
25	0.999	0.7525	0.750	0.750	25	0.170	0.3332	0.750	0.926

Table 4
Optimal allocation for different market prices of risk

In the table, T is the time horizon of the investor. LW is the value of the human capital wealth of the investor. π^L is the dollar amount invested in the risky security in excess of the optimal allocation in Merton's benchmark. π^M is the optimal allocation in Merton's benchmark. $\pi = \pi^L + \pi^M$ is the total allocation in the risky security. X is financial wealth, so that $\pi/(X + LW)$ is the proportion of total wealth (financial plus human capital) invested in the risky security. δ is the parameter that measures the ability of the investor. μ is the weight of intertemporal consumption in the utility function of the investor. Other parameter values of the model in this table are as follows: $\gamma = -5$, so that risk aversion $1 - \gamma = 6$; the upper bound in the effort is $K_2 = 0.2$; the weight of the utility of final wealth in total utility is $\lambda = 0.7$; interest rate is $r = 0.035$; the ability of the investor is $\delta = 0.3$; the discount factor for the cost of effort is $r_u = 0.05$; the subjective discount factor of the utility of intertemporal consumption is $r_c = 0.05$; finally, initial liquid wealth is $X = 1$.

$\theta = 0.3$

$\mu = 0.1$					$\mu = 0$				
T	LW	π^L	π^M	$\frac{\pi}{X+LW}$	T	LW	π^L	π^M	$\frac{\pi}{X+LW}$
1	0.059	0.0147	0.250	0.250	1	0.040	0.0662	0.250	0.304
5	0.275	0.0688	0.250	0.250	5	0.068	0.1175	0.250	0.344
10	0.506	0.1266	0.250	0.250	10	0.049	0.0854	0.250	0.320
15	0.700	0.1750	0.250	0.250	15	0.028	0.0479	0.250	0.290
20	0.863	0.2157	0.250	0.250	20	0.013	0.0231	0.250	0.273
25	1.000	0.2499	0.250	0.250	25	0.006	0.0102	0.250	0.259

$\theta = 0.2$

$\mu = 0.1$					$\mu = 0$				
T	LW	π^L	π^M	$\frac{\pi}{X+LW}$	T	LW	π^L	π^M	$\frac{\pi}{X+LW}$
1	0.059	0.0098	0.167	0.167	1	0.040	0.0458	0.167	0.204
5	0.275	0.0459	0.167	0.167	5	0.067	0.0782	0.167	0.229
10	0.506	0.0844	0.167	0.167	10	0.047	0.0550	0.167	0.222
15	0.700	0.1167	0.167	0.167	15	0.025	0.0295	0.167	0.196
20	0.863	0.1438	0.167	0.167	20	0.011	0.0134	0.167	0.180
25	1.000	0.167	0.167	0.167	25	0.005	0.0055	0.167	0.171