

Beliefs Regarding Fundamental Value and Optimal Investing ^{*}

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Abstract

Standard optimal portfolio selection models take no account of the special information that active investors believe they possess. For example, active investors who believe they can place bounds on the price of a security will want to use that information when assessing risk and expected return in order to construct an optimal portfolio. In this paper, we use two continuous-time models to analyze how placing boundaries on the price of a stock affects assessed risk, expected returns, and the optimal holdings of an active investor, and how those vary as a function of the relation between the stock price and the boundaries. In particular, the optimal strategy takes significant long/short positions as the price nears its lower/upper boundary.

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JEL Classification: G11

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1 Introduction

From the standpoint of an active investor, there is a fundamental paradox with respect to the application of standard portfolio selection models. To model the relation between risk and expected returns, those models employ restrictive assumptions. The most stringent of these assumptions, employed by most standard models, is that investors have homogeneous beliefs defined over investments opportunities whose variations are determined by stationary stochastic processes.

The paradox is that the reason for being an active investor is the conviction that some combination of price volatility, non-stationarity, incomplete and asymmetric information, and heterogeneous information processing capabilities, lead to inefficient pricing that can be exploited¹. The fact that standard models cannot be employed directly has been recognized in various contexts by numerous professional investors. For example, as reported by Penman (2007), famed investor Warren Buffett puts the matter this way: *“The CAPM says that if the price of a stock drops more than the market, it has a high beta: It’s high risk. But if the price goes down because the market is mispricing the stock relative to other stocks, then the stock is not necessarily high risk: The chance of making an abnormal return has increased, and paying attention to fundamentals makes the investor more secure, not less secure.”*² In a similar vein, Morningstar (2004), a leading provider of investment analysis, states that, *“In deciding the rate to discount future cash flows, we ignore stock-price volatility (which drives most estimates of beta) because we welcome volatility if it offers opportunities to buy a stock at a discount to its fair value. Instead, we focus on the fundamental risks facing a company’s business.”*

Both Buffet and Morningstar are making the same basic point. If markets are sometimes inefficient, and if the active investors believe they can exploit that inefficiency then it makes little sense to rely on measures of risk derived standard models that ignore the perceived inefficiency. To be useful to active investors, models must be extended to incorporate what they believe is their fundamental information. One problem in this regard is that the method of incorporation depends on the precise nature of the inefficiency that the investor believes exists.

In this paper, we consider an investor who has beliefs about the range of the possible values that a particular risky asset can take. These beliefs may stem from some superior information the investor has, which, in principle, should be updated in a Bayesian way. However, we abstract from such updating and direct modeling of the information. Instead,

¹Friedman (1953) argues that security prices reflect fundamental values. Otherwise, if securities are mispriced, as a result of irrational investors’ behavior, rational investors will take advantage of the mispricing, and therefore push prices to their fundamentals. Other researchers, on the other hand, argue that security prices may persistently diverge from their fundamentals. Shiller (1984), De Long et al. (1990) argue that when the transaction costs exceed potential profits then the prices may not converge to their fundamentals. Campbell and Shiller (2001) show that deviations from the fundamentals have provided valuable forecasting information for future stock prices. When the Price/Earnings ratio has been above its historical mean, stock prices tended to fall. Likewise, when the Dividends/Price ratio has been above its mean, stock prices tended to rise.

²This is consistent with the Fama and French (1992) observation that Book-to-Market (B/M) ratios are positively correlated with subsequent stock returns, a relation that is known as the book-to-market effect. That is, value stocks - the stocks with higher B/M ratios yield higher returns compared to growth stocks - stocks with lower B/M ratios.

we consider the model in its reduced form, in which the extra information has been transformed into the beliefs on the range of the risky asset. Given such beliefs, the investor is motivated to be partially exposed to the specific risk of the risky asset, while keeping the rest of the portfolio diversified.

More specifically, we consider two models, similar in spirit. In the first, the investor believes that he can determine fundamental bounds on the value of the asset during the time interval, $[0, T]$. If the market price penetrates either of those boundaries, it reverts back toward the boundary³. We refer to this first case as range reversion or RR⁴.

In the second, related case, the active investor believes that as of time $t = 0$ he can place strict bounds on the distribution of the asset price at time T . As a result, rather than being lognormal, as in the Black-Scholes framework, the time- T distribution of the stock price is truncated at the boundaries. This case is referred to as range distribution, or RD⁵. In both cases, it is intuitively clear that the active investor will perceive that both the risk and expected return of the stock will be a function of the current price.⁶

The fact that risk and return vary with price, from the standpoint of the active investor, is an extension of the classical asset pricing models such as CAPM⁷ in which risk and expected return are not correlated with the price of a security. In our models as the price approaches a boundary the impact on risk and expected return grows. For example, as both Morningstar and Mr. Buffett suggest, as the price nears its lower bound risk falls and expected return increases. Naturally, this affects the optimal portfolio of the active investor, who would increase his holdings of the security under such circumstances. This is contrary to stochastic volatility models in which volatility increases as prices drop, causing the investor to reduce his holdings.

The framework for our analysis is one in which the investor allocates his wealth between a risky stock (about which the investor has superior information and forms beliefs based on that information), a market index and a risk-free asset. Within this framework, we consider two alternative scenarios. The first is a static mean-variance scenario in which the portfolio holdings are set at time 0 and cannot be altered. The second is a dynamic optimization scenario in which the portfolio holdings can be adjusted continuously.⁸

The paper proceeds as follows. First we provide a mathematical discussion of both the

³As Campbell and Shiller (2001) observe, the mean reversion in valuation ratios and the ability of the ratios to forecast future prices are not new concepts. Mean reversion in fundamentals has been frequently discussed as a forecasting tool for price movements over the last century.

⁴This model accommodates the possibility that stock prices can diverge from their fundamentals for prolonged periods. The evidence in support of this proposition dates at least to Summers (1986).

⁵Liu and Longstaff (2004) study a related problem. In their setting, an investor has perfect knowledge about the future value of the security at time T , and thus faces an arbitrage opportunity related to a security, which is modeled as a Brownian Bridge process.

⁶This result is consistent with the 3-factor pricing model of Fama and French (see Fama and French (1993)) in which the factors HML or SMB are functions of the price of the security.

⁷see Sharpe (1964, 1970), Lintner (1965a, 1965b), and Mossin (1966).

⁸Black and Litterman (1990) were one of the first to use investors' subjective views along with CAPM as a benchmark in portfolio management context. Their model uses a Bayesian approach to combine the subjective views of an investor regarding the expected returns of assets with the market equilibrium expected returns (via the prior distribution) to form a new estimate of expected returns. The resulting new set of returns (the posterior distribution) leads to portfolios with more stable weights and thus results in a better asset allocation. A continuous-time version of this approach is applied in Cvitanić, Lazrak, Martellini and Zapatero (2006).

range reversion and range distribution models. Straightforward proofs are omitted, and are available from the authors. In situations where there are no closed form solutions to the problems, we provide some results via Monte Carlo simulations. Based on the simulations, we demonstrate how risk, expected return, and optimal portfolio holdings vary as a function of the security price for both the range reversion and range distribution models. Second, we contrast our results with those that emerge from standard models, and compute how much an investor loses in expected utility when following a wrong model. Finally, we finish with some conclusions and possible extensions.

2 RR Model: “Range Reversion”

The two models we consider are similar in spirit. In the first model, called range reversion (RR), the investor’s information regarding the fundamental value of a risky asset leads him to believe that although the price can go outside of a specific range of values, it will tend to revert back to that range with certain speed of reversion. In the second model, called range distribution (RD), the asset price at the final time takes values strictly inside the range with probability one. Our results are qualitatively the same for both models⁹, but there are some important differences:

- in RR, the volatility of the asset is fixed, but the drift varies stochastically depending on the stock price, and the variation in the risk-premium is driven by the drift; in RD, the risk premium is fixed, while the drift and the volatility vary stochastically (in a fully correlated fashion).

- in RR, the distribution of the asset price is not drivable in closed form. Thus, it is difficult to construct specific equilibrium models in which such a process would arise; in RD the distribution of the final value of the risky asset is known, and, in an equilibrium model with no intermediate consumption, it could be interpreted as the exogenously given value of the final dividend.

Using the RD model it is possible to use a wide range of stock price distributions at a fixed future time T , while maintaining a constant risk premium (and it can be extended to stochastic risk premium). In this paper, we only study the RD model for the case of a truncated Log-normal distributions.

2.1 Details of the RR model

In the range reversion model the active investor believes he can determine an approximate lower bound L_t and an approximate upper bound U_t of a risky asset (sa a stock) with price S_t . The investor believes that the price will revert back into the range $[L_t, U_t]$ with a certain speed over a given time interval $[0, T]$. We allow L_t and U_t to vary over time. This type of modeling is similar to the traditional mean-reversion models of prices, except in those models the stock prices mean-revert to a single value, whereas in our case the prices mean-revert

⁹The second model is not a limiting case of the first when the speed of reversion increases, because in the limit we get a process which reflects back from the boundaries of the range, rather than a process which never hits the boundaries.

within time-dependent boundaries. Consequently, mean reversion models are a special case of our model in which $L_t = U_t$.

For Monte Carlo computations of the results, it is more convenient to work with log-values. Thus, we model the range reversion of prices as follows. Denote

$$\tilde{L}_t = \log L_t, \quad \tilde{U}_t = \log U_t, \quad Y_t = \log S_t.$$

In this framework, the log-asset price Y_t is given by the Stochastic Differential Equation (SDE)

$$dY_t = \left[\mu_t - \sigma_t^2/2 + n_L \max\{0, \tilde{L}_t - Y_t\} - n_U \max\{0, Y_t - \tilde{U}_t\} \right] dt + \sigma_t dB_t. \quad (2.1)$$

Here, B_t is a one-dimensional standard Brownian Motion process, and n_L and n_U are the “speeds of reversion” from the lower and upper bounds respectively. If the stock price S is below L (log-stock price Y is below \tilde{L}) the term containing n_L pushes the price up towards L . Similarly if S is above U it reverts back towards U at a speed n_U . Setting $\tilde{L} = -\infty$, $\tilde{U} = \infty$ gives the standard generalized Black-Scholes-Merton model, henceforth BSM model. In the computations, we will take μ, σ to be deterministic and constant.

2.2 Optimal investing in the RR model

In this section, we compare optimal investments of a passive investor who follows the BSM model with an active investor who believes in the RR model. Consider a BSM model for the stock price given by

$$\hat{S}_t = \hat{S}_0 e^{(\hat{\mu} - \hat{\sigma}^2/2)t + \hat{\sigma} B_t}.$$

Also assume that there exists a market index I , the dynamics of which follow a geometric Brownian Motion process:

$$I_t = I_0 e^{(\mu_I - \sigma_I^2/2)t + \sigma_I [\rho B_t + \sqrt{1 - \rho^2} W_t]}$$

where W is a Brownian Motion process, independent of B . Moreover, assume that there is a risk-free asset with constant interest rate $r \geq 0$. We assume that both the BSM investor and the RR investor have the same beliefs regarding the index I dynamics and the risk-free asset. They differ in their beliefs about the dynamics of S only. For the given time horizon $T > 0$, we assume that in the RR model (2.1) $\sigma_t = \sigma$, $\mu_t = \mu$ are constants and that

$$\tilde{L}_t = \tilde{L} - r(T - t), \quad \tilde{U}_t = \tilde{U} - r(T - t)$$

where \tilde{L}, \tilde{U} are constants. In other words, the likely range for the final stock price S_T at time T is $[e^{\tilde{L}}, e^{\tilde{U}}]$, and the likely range for the stock prices S_t for $t < T$ is $[e^{\tilde{L}} e^{-r(T-t)}, e^{\tilde{U}} e^{-r(T-t)}]$.

As the measure of risk we compute the beta of asset S^{10} , defined by

$$\beta = \frac{Cov[I_T/I_0, S_T/S_0]}{Var[I_T/I_0]}. \quad (2.2)$$

¹⁰We only consider here partial equilibrium, in which investors take prices as given and cannot influence them; thus, it is not a priori clear that beta is the right measure of risk in this model.

In order to simplify the calculations and solve the portfolio optimization problem, we assume that the investors have logarithmic preferences. In this framework, we compute optimal investment allocations for both static, Markowitz Mean-Variance portfolios, and for dynamic, Merton Log-optimal portfolios.

More specifically, denote the value of the investor's portfolio at time T by X_T . For the Markowitz Mean-Variance setting, we consider the maximization of

$$E \left[\frac{X_T}{X_0 e^{rT}} - 1 - \frac{\gamma}{2} \left(\frac{X_T}{X_0 e^{rT}} - 1 \right)^2 \right]. \quad (2.3)$$

In particular, with the risk aversion parameter γ equal to one, this approximates to $\log(X_T/X_0) = \log(1 + (\frac{X_T}{X_0 e^{rT}} - 1))$. For the Merton case, we maximize $E[\log(X_T)]$.

2.2.1 Optimal mean-variance investment

To compare our results with those of the standard model, we denote by $\pi = (\pi_S, \pi_I)$ the optimal weight proportions in assets S and I respectively, in the RR model. The optimal weights for the BSM investor are denoted by, $\hat{\pi} = (\hat{\pi}_S, \hat{\pi}_I)$. We place a superscript M , π^M for the static, Markowitz case, and no superscript for the dynamic, Merton case. Finally, in order to simplify notation, we define the quantities representing returns in excess of the risk free rate,

$$\bar{I}_T = I_T - I_0 e^{rT}, \quad \bar{S}_T = S_T - S_0 e^{rT}. \quad (2.4)$$

In this framework, we first derive formulas for optimal investment strategies, and later compare the numerical results for the two models. The following is a familiar result for mean-variance optimization that we use in our analysis.

Proposition 2.1 *For an investor maximizing expected value of the expression in (2.3), the optimal static portfolio proportions are given by the formulas:*

$$\pi_S^M = \frac{2 - \gamma}{\gamma} S_0 e^{rT} \frac{E[\bar{S}_T]E[\bar{I}_T^2] - E[\bar{I}_T \bar{S}_T]E[\bar{I}_T]}{E[\bar{I}_T^2]E[\bar{S}_T^2] - (E[\bar{S}_T \bar{I}_T])^2} \quad (2.5)$$

$$\pi_I^M = \frac{2 - \gamma}{\gamma} S_0 e^{rT} \frac{E[\bar{I}_T]E[\bar{S}_T^2] - E[\bar{S}_T \bar{I}_T]E[\bar{S}_T]}{E[\bar{I}_T^2]E[\bar{S}_T^2] - (E[\bar{S}_T \bar{I}_T])^2} \quad (2.6)$$

and similarly for $\hat{\pi}_S^M$ and $\hat{\pi}_I^M$, which are just like the above expressions with $\bar{\hat{S}}_T$ used instead of \bar{S}_T .

Proof: This well-known result, that is obtained by the standard mean-variance optimization techniques, implies that the optimal portfolio is proportional to the inverse of the variance-covariance matrix calculated using excess returns:

$$\left\{ \begin{array}{cc} E[\bar{S}_T^2] & E[\bar{S}_T \bar{I}_T] \\ E[\bar{S}_T \bar{I}_T] & E[\bar{I}_T^2] \end{array} \right\}^{-1} (E[\bar{S}_T], E[\bar{I}_T])^{Tr}$$

■

The main reason why the optimal strategies qualitatively differ between the RR and the BSM model is that in the BSM models the expected relative return is given by

$$E[\hat{S}_T/S_0] = e^{\hat{\mu}T}$$

which is independent of the current stock price. This is not the case for the RR model, where the expected return and the beta depend on the stock price relative to the boundaries. In the Comparative Statics subsection below, we provide numerical results which illustrate the impact of the stock price on risk and expected return.

2.2.2 Optimal dynamic investment

Next, we consider vectors of proportions $\pi, \hat{\pi}$ that are optimal for the investor who can continuously rebalance his portfolio (without transaction costs or other frictions).

For the RR model with constant volatility σ ,

$$dS_t = \mu_t^S S_t dt + \sigma_t^S S_t dB_t$$

where

$$\begin{aligned} \sigma_t^S &= \sigma, \\ \mu_t^S &= \mu_t + n_L \max\{0, \tilde{L}_t - \log S_t\} - n_U \max\{0, \log S_t - \tilde{U}_t\}. \end{aligned}$$

Introduce the volatility matrix,

$$\Sigma_t = \begin{Bmatrix} \sigma_t^S & 0 \\ \rho\sigma_I & \sqrt{1 - \rho^2}\sigma_I \end{Bmatrix},$$

and the vector of expected excess return rates,

$$R_t = \{\mu_t^S - r, \mu_I - r\}^{11}.$$

Then, we know from the classical Merton's problem (see Merton (1969, 1971)), that the optimal vector of proportions π of wealth to be held in S and I , for an investor with logarithmic utility, is given by

$$\pi_t = (\Sigma_t \Sigma_t^{Tr})^{-1} R_t. \quad (2.7)$$

In the same fashion, we can solve for the optimal $\hat{\pi}_t$ in the BSM model with μ^S, σ^S replaced by $\hat{\mu}, \hat{\sigma}$. We use (2.7) to compute optimal dynamic portfolios. Again, the main difference between the two models is that the parameters entering the computation in (2.7) are independent of the stock price for the BSM model, while this is not the case for the RR model.

Remark 2.1 It would be possible also to compute the optimal portfolio holdings in the case of power utility functions, but it requires using relatively involved numerical methods. Because the opportunity set (here the drift) is stochastic, there would be a hedging demand in the portfolio. That is, the portfolio would differ from the myopic holding, which is equal to the expression in (2.7) divided by the risk aversion coefficient. However, our main goal is not to measure the hedging demand but to show the qualitative difference between the RR and BSM models, and that is sufficiently well illustrated by the case of the logarithmic utility and mean-variance preferences.

¹¹Notice that Σ_t does not depend on t .

2.2.3 Comparative Statics

To assess the economic significance of the difference between the RR and BSM models we use the following benchmark parameters, which approximate the actual return parameters observed for individual common stocks and a typical market index:

$$T = 1, \quad r = 0.01, \quad I_0 = 1, \quad S_0 = 1, \quad X_0 = 1, \quad \mu = 0.07, \quad \sigma = 0.21, \quad \mu_I = 0.06, \quad \sigma_I = 0.17, \quad \rho = 0.2,$$

$$\tilde{U} = \log S_0 + \mu T + 2\sigma\sqrt{T}, \quad \tilde{L} = \log S_0 + \mu T - 2\sigma\sqrt{T}.$$

The risk-free rate is taken to be the average real risk-free rate over the last 50 years, and the mean and the variance of the stock and the index returns are also the averages of actual means and variances over the last 50 years.

We choose $\hat{\mu}$, $\hat{\sigma}$ so that the mean and the variance of the stock price at time T are the same in the two models, for $S_0 = \hat{S}_0 = 1$. That is, we choose $\hat{\mu}$ and $\hat{\sigma}$ so that

$$E[S_T] = E[\hat{S}_T] = e^{\hat{\mu}T},$$

$$E[S_T^2] = E[\hat{S}_T^2] = e^{(2\hat{\mu} + \hat{\sigma}^2)T}.$$

In the calculations, we keep $\hat{\mu}$ and $\hat{\sigma}$ fixed at the initial level, corresponding to $S_0 = 1$, but allow S_0 to vary. The interpretation of this procedure is as follows. Suppose the stock price is low today. For the BSM investor (since BSM is a stationary model) this does not change the long-run mean $\hat{\mu}$ and volatility $\hat{\sigma}$ (and thus the risk-premium on the stock). Consequently, the optimal investment weights will not change with the stock price. For the RR investor, however, if the stock price is close to what the active investor believes is its approximate lower bound, the risk premium should be higher (because the expected returns would be higher and beta would be lower) than predicted by BSM model, and the optimal weight should be greater. Table 1 presents the results of the calculations comparing the RR model with the BSM model.¹² Panel A of Table 1 shows that the investment weights for the RR investor change markedly as the stock price approaches either of the bounds. When the stock is near the bottom boundary the RR investor greatly increases his relative holding in the stock and the reverse is true for the upper boundary¹³. In order to compensate for this behavior, the RR investor also takes more extreme positions in the index than the BSM investor, but not as extreme as in the stock, because the index has unlimited range in the model¹⁴. Panel B of Table 1 shows the Merton's optimal dynamic portfolio weights for the

¹²The portfolio weights for the RR model are actually expected values of the portfolio weights at time $t = T/2$. We do this because at time $t = 0$ the stock price is inside the believed bounds, and not much interesting happens.

¹³This may help explain why fund managers may want to hold less-diversified portfolios. A fund manager that has superior skills or information which allow him to estimate boundaries, may decide to hold a more concentrated portfolios. Coval and Moskowitz (1999, 2001) show that mutual funds have strong preferences for investing in local firms where they might have informational advantages. Our analysis is consistent with those results.

¹⁴These findings may be used to explain the investment strategies of corporate executives. Under the assumption that executives are more knowledgeable about their company's future prospects, they can better assess the "closeness" of their company's stock price to fundamental bounds and make investment decisions based on that assessment. For example, if an executive believes his firm's stock is close to the lower bound (or is undervalued relative to the assessed fundamentals), he will hold more shares and thus, be under-diversified.

Log-investors of the RR type and the BSM type. The quantitative results are even more dramatic than those for the mean-variance calculation, but the qualitative behavior is the same.

3 RD Model: Modeling return distribution at a future time

We now analyze a class of models in which the active investor believes he has superior information regarding the future stock price distribution, at a fixed future time T . For example, this is consistent with a situation in which the investor believes that the stock price will be constrained to be in the vicinity of its fundamental value in the long run. We refer to this scenario as the RD model¹⁵. As mentioned previously, this model is conceptually similar to the RR model – they both reflect the idea that the investor has strong beliefs on the fundamental value of the investment asset, which translate into bounds on its price. Even though the RR and RD models are similar in spirit, both models are studied separately as one cannot get one model as a limiting or a particular case of the other.

Let S_T be a random variable which represents the active investor's beliefs regarding the distribution of the stock price at time T . We assume that S_T is of the form

$$S_T = f(B_T^Q)$$

where f is a deterministic function and B^Q is a Brownian Motion process, but under a probability measure Q , which is risk-neutral from the point of view of the active investor. Thus, the Brownian Motion process B under the actual physical probability is such that

$$B^Q(t) = B(t) + \theta^Q t$$

where the risk premium θ^Q perceived by the investor is assumed to be constant. From this investor's point of view, the stock price is the expected value of its discounted future value under the probability measure Q . That is, we have

$$S_t = E_t^Q [e^{-r(T-t)} f(B_T^Q)]. \quad (3.1)$$

Remark 3.1 If the investor believes that $S_T = f(B_T^Q)$, then it follows from the Feynman-Kac theorem that $S_t = V(t, B_t^Q)$, where the function V is a solution to the P.D.E. $\partial_t V + \frac{1}{2} \partial_{xx} V - rV = 0$ with the boundary condition $V(T, x) = f(x)$, $\forall x$.

The focus of the next subsection is on the modeling of the stock price at a fixed future time, and on the calibration of the risk premium θ^Q to the current stock price using (3.1) with $t = 0$. Put differently, because the active investor believes in a different distribution than the BSM investor, he also believes in a different risk premium, hence in a different risk-neutral probability.

¹⁵Liu and Longstaff (2004) study investment decisions of an active investor who faces an arbitrage opportunity regarding the future value of a security about which the investor has perfect foresight.

Remark 3.2 This approach provides a flexible way of modeling the distribution of the stock price at a specific future time, and thereby provides flexibility in modeling the risk premium. Here, we have chosen the risk premium to be constant, however, the same approach works with an arbitrary model of the risk premium (under the constraint (3.1) at $t = 0$).

3.1 BSMCT Model

To make the analysis tractable, we study a special case of RD models, in which the log-price $Y_T = \log S_T$ has a conditionally truncated normal distribution. More precisely, the distribution of $Y = Y_T$ is the distribution of the normal random variable with mean

$$m = \log S_0 + (\mu - \sigma^2/2)T \quad (3.2)$$

and variance

$$\tilde{\sigma}^2 = \sigma^2 T$$

conditional on taking values in the interval with endpoints

$$\tilde{L} = \log L, \quad \tilde{U} = \log U.$$

We call this model BSMCT (for Black-Scholes-Merton Conditionally Truncated). It is known that the distribution function of Y is then given by

$$F_Y(y) = \frac{N\left(\frac{y-m}{\tilde{\sigma}}\right) - N\left(\frac{\tilde{L}-m}{\tilde{\sigma}}\right)}{N\left(\frac{\tilde{U}-m}{\tilde{\sigma}}\right) - N\left(\frac{\tilde{L}-m}{\tilde{\sigma}}\right)}.$$

where $N(\cdot)$ is the standard normal cumulative distribution function. Computing F_Y^{-1} from the preceding formula and using the fact that for an arbitrary continuous distribution function F , $S_T = F^{-1}(N(B_T/\sqrt{T}))$ will have F as its distribution, we find that the corresponding model for the stock price at time T is

$$\log(S_T) = m + \tilde{\sigma} N^{-1} \left\{ N\left(\frac{\tilde{L} - m}{\tilde{\sigma}}\right) + N\left(\frac{B_T^Q - \theta^Q T}{\sqrt{T}}\right) \left[N\left(\frac{\tilde{U} - m}{\tilde{\sigma}}\right) - N\left(\frac{\tilde{L} - m}{\tilde{\sigma}}\right) \right] \right\}. \quad (3.3)$$

Recall that θ^Q is not a free parameter – it has to be chosen so that

$$S_0 = E^Q[e^{-rT} S_T].$$

Denoting

$$U^* = \frac{\tilde{U} - m}{\tilde{\sigma}}, \quad L^* = \frac{\tilde{L} - m}{\tilde{\sigma}},$$

the formulas needed for computations are based on the following proposition.

Proposition 3.1 *In the BSMCT model, the stock price at time T can be written as*

$$S_T = e^{m + \tilde{\sigma} N^{-1} \left\{ N(L^*) + N\left(\frac{B_T^Q - T\theta^Q}{\sqrt{T}}\right) [N(U^*) - N(L^*)] \right\}}$$

and the stock price for $t < T$ is given by $S_t = V(t, B_t^Q)$ where

$$V(t, x) = e^{-r(T-t)} E^Q \left[e^{m + \tilde{\sigma} N^{-1} \left\{ N(L^*) + N \left(\frac{x + B_T^Q - B_t^Q - \theta^Q (T-t)}{\sqrt{T-t}} \right) [N(U^*) - N(L^*)] \right\}} \right]. \quad (3.4)$$

We also have

$$V_x(0, 0) = e^{-rT} E^Q \left[\frac{\tilde{\sigma}}{\sqrt{T}} S_T \frac{n \left(\frac{B_T^Q - T\theta^Q}{\sqrt{T}} \right) [N(U^*) - N(L^*)]}{n \left(\frac{\log S_T - m}{\tilde{\sigma}} \right)} \right]. \quad (3.5)$$

Proof: We get the expression for S_T from (3.3), we get (3.4) from (3.1), and (3.5) from differentiating (3.4). ■

Given that

$$dS_t = S_t[\mu_t^S dt + \sigma_t^S dB_t],$$

it follows that in the terminology of CAPM the “instantaneous beta” of asset S is given by

$$\beta_S(t) = \frac{\rho \sigma_t^S}{\sigma_I} \quad (3.6)$$

where σ_I is the volatility of the index. In order to compute the instantaneous beta of S we need the following lemma, which follows directly from Ito’s rule.

Lemma 3.1 *In BSMCT model, we have*

$$\sigma_t^S = \frac{V_x(t, B_t^Q)}{S_t}.$$

3.2 Optimal investing in the BSMCT model

In this section, we compare optimal investments of an investor who uses a BSM model or an RR model, and an investor who uses a BSMCT model. The BSM model for S is the same as in the previous section. As before, we also assume that both the BSM investor and the BSMCT investor have the same beliefs regarding the index I and the risk-free asset, but they differ in their beliefs about S . Note that since S_T takes values in $[L, U]$, the stock price values prior to time T are contained in the corresponding “discounted” interval $[Le^{-r(T-t)}, Ue^{-r(T-t)}]$. We assume again that the investor has a logarithmic utility and we compute optimal investment weights for a static, Markowitz mean-variance portfolio, and for a dynamic, Merton Log-optimal portfolio. The qualitative behavior of the static portfolios are the same as for the dynamic portfolios, but they differ quantitatively - the latter being more variable than the former.

3.2.1 Mean-Variance portfolio

The formulas here are the same as in the RR case. From Proposition 2.1, it follows that

$$\pi_S^M = \frac{2 - \gamma}{\gamma} S_0 e^{rT} \frac{E[\bar{S}_T] E[\bar{I}_T^2] - E[\bar{I}_T \bar{S}_T] E[\bar{I}_T]}{E[\bar{I}_T^2] E[\bar{S}_T^2] - (E[\bar{S}_T \bar{I}_T])^2} \quad (3.7)$$

$$\pi_I^M = \frac{2 - \gamma}{\gamma} S_0 e^{rT} \frac{E[\bar{I}_T] E[\bar{S}_T^2] - E[\bar{I}_T \bar{S}_T] E[\bar{S}_T]}{E[\bar{I}_T^2] E[\bar{S}_T^2] - (E[\bar{S}_T \bar{I}_T])^2} \quad (3.8)$$

and similarly for $\hat{\pi}_S^M$ and $\hat{\pi}_I^M$, which are just like the above expressions with \bar{S}_T used in the above formulas instead of \hat{S}_T .

3.2.2 Dynamic portfolios

We know from Proposition 3.1 (see Remark 3.1) that $S_t = V(t, B_t^Q)$ and also that

$$dS_t = rS_t dt + V_x(t, B_t^Q) dB_t^Q.$$

It follows from the above that

$$dS_t = \mu_t^S S_t dt + \sigma_t^S S_t dB_t$$

with

$$\mu^S = r + \theta^Q V_x / S, \quad \sigma^S = V_x / S.$$

As mentioned earlier, in this model the drift and the volatility of the asset are stochastic, but perfectly correlated, so that the risk premium is constant. Define the volatility matrix,

$$\Sigma = \left\{ \begin{array}{cc} \sigma^S & 0 \\ \rho \sigma_I & \sqrt{1 - \rho^2} \sigma_I \end{array} \right\},$$

and the vector of excess returns,

$$R = \{\mu^S - r, \mu_I - r\}.$$

Then, from the classical Merton problem, we know that the vector of optimal proportions, π_t , of wealth to be held in the stock and in the index for an investor with logarithmic utility is given by

$$\pi = (\Sigma \Sigma')^{-1} R. \quad (3.9)$$

Similarly, the vector of optimal proportion, $\hat{\pi}$, in the Black-Scholes model is calculated by replacing μ^S, σ^S with $\hat{\mu}, \hat{\sigma}$ in the above formula (3.9). We use (3.9) to compute optimal dynamic portfolios.

3.3 Comparative Statics

In the numerical computations, we again use these benchmark parameters as in the previous section:

$$T = 1, \quad r = 0.01, \quad I_0 = 1, \quad S_0 = 1, \quad X_0 = 1, \quad \mu = 0.07, \quad \sigma = 0.21, \quad \mu_I = 0.06, \quad \sigma_I = 0.17, \quad \rho = 0.2,$$

$$L = 0.59, \quad U = 1.97, \quad \gamma = 1.$$

We choose $\hat{\mu}$, $\hat{\sigma}$ so that the mean and the variance of the stock at time T are the same in the two models, for $S_0 = \hat{S}_0 = 1$. That is, we choose $\hat{\mu}$ and $\hat{\sigma}$ so that

$$E[S_T] = E[\hat{S}_T] = e^{\hat{\mu}T},$$

$$E[S_T^2] = E[\hat{S}_T^2] = e^{(2\hat{\mu} + \hat{\sigma}^2)T}.$$

As before, we keep $\hat{\mu}$ and $\hat{\sigma}$ fixed at the level corresponding to $S_0 = 1$, but vary the initial stock price S_0 by varying the risk premium θ^Q in this case.

We consider a case of the BSMCT model, in which m changes with S_0 . This corresponds to a case of an active investor whose belief about S_T changes with the current value of S_0 . This belief about S_T is close to the one of the BSM investor, except that the distribution of the log-price for the BSMCT investor is a conditionally truncated normal.¹⁶

Figure 1 shows how the risk premium θ^Q varies with the stock price for the BSMCT model. As the stock price rises, the risk premium declines, even becoming negative at very high stock prices. Once again, we have a model in which the risk, the expected return (and the risk premium) change with the stock price.

Panel A of Table 2 presents expected returns $E[S_T/S_0]$ for both non-standard models. The expected returns in the RR case are less sensitive to the stock price than in the BSMCT model because the bounds in that model are soft bounds. The expected returns are negatively correlated with initial stock prices.

Panel B of Table 2 presents the variances $Var[S_T/S_0]$ of returns. Here too, the variances are more sensitive to the stock price in the RR model than in the BSMCT model. In both cases, the variance is lower at the low and high levels of stock prices and higher at mid-levels. The stock is least risky when the price is close to the boundaries.

Figure 2 presents the results for the static betas as defined in (2.2), which mirror the results for the variances across the different models.

On the other hand, the instantaneous betas in Figure 3, as defined in (3.6), are quite different from the static betas. In the BSCT model: the instantaneous betas are low when the stock price is close to the lower and upper bounds, and higher in the middle, but they are always lower than the stationary beta in the BSM model. This shows that because of his perceived fundamental information the active investor will always interpret stocks about which he has information as less risky. The greater variation in beta as a function of the

¹⁶We also considered two other cases but do not report the results here: 1) a case in which m was a constant, fixed at $S_0 = 1$. This would be the case of fixed beliefs about the distribution of the stock price at time T ; 2) a case in which m is fixed, but corresponds to the case $L = 0$ and $U = \infty$. That is, there are no bounds on the distribution of S_T , as in the BSM model, but the investor does not adapt his beliefs about S_T as the value of S_0 changes, unlike in the BSM model. In both models the results are qualitatively similar to the model we study here.

stock price is due to the fact that this is a local beta, which does not take into account possible future changes of the stock price dynamics, and the fact that the local volatility is lower when closer to the bounds. The instantaneous beta of the RR model is not presented, but it is the same as the stationary BSM beta, since the volatility is fixed in the RR model.

Table 3 presents optimal Markowitz weights in the stock and the index for the BSMCT model. The investment weights vary with the stock price in a fashion similar to the RR model. As the stock price approaches the upper boundary, the investor sells the stock short in large amounts (-471 percent) and invests almost 100 percent of the wealth in the index. At the lower levels of the stock price, the investor puts 280 percent of the wealth in the stock as it is perceived to offer high expected return and less risk as shown in Table 2 and Figures 2 and 3.

4 Utility Gains and Losses across models

In this section we compare the performance of static Markowitz investment strategies that follow from the two proposed models. The benchmark for comparison in the first case is a situation in which the true model of stock price movements is the RR model, and the investor follows the RR model. The benchmark for comparison in the second case is the situation in which the true model of the stock price movements is the BSM model, and the investor follows the BSM model. In both cases, we compare the certainty equivalents of the utility of terminal wealth of an investor that follows a “wrong” model (the perceived stock price dynamics are different from the true dynamics), with an investor that follows the benchmark.

The wealth-process of the investor at time T is given by

$$X_T(S, \delta_S, \delta_I) = \delta_I I_T + \delta_S S_T + (X_0 - \delta_I I_0 - \delta_S S_0)e^{rT} \quad (4.1)$$

where δ_I and δ_S are computed using (2.4):

$$\delta_S S_0 / X_0 = \frac{2 - \gamma}{\gamma} S_0 e^{rT} \frac{E[\bar{S}_T]E[\bar{I}_T^2] - E[\bar{I}_T \bar{S}_T]E[\bar{I}_T]}{E[\bar{I}_T^2]E[\bar{S}_T^2] - (E[\bar{S}_T \bar{I}_T])^2} \quad (4.2)$$

$$\delta_I I_0 / X_0 = \frac{2 - \gamma}{\gamma} S_0 e^{rT} \frac{E[\bar{I}_T]E[\bar{S}_T^2] - E[\bar{I}_T \bar{S}_T]E[\bar{S}_T]}{E[\bar{I}_T^2]E[\bar{S}_T^2] - (E[\bar{S}_T \bar{I}_T])^2}. \quad (4.3)$$

Using the dynamics of S and I , and the above formulas with $\gamma = 1$, we can compute the expected utility of the investor

$$u(S, \delta_S, \delta_I) := E \left[\frac{X_T(S, \delta_S, \delta_I)}{X_0 e^{rT}} - 1 - \frac{1}{2} \left(\frac{X_T(S, \delta_S, \delta_I)}{X_0 e^{rT}} - 1 \right)^2 \right]. \quad (4.4)$$

In this context, we study the performances of our models by examining the certainty equivalents (CE) of the models and comparing them to the benchmark case. We do this analysis for the case of Markowitz’ static portfolios. Define the following ratios in the Markowitz’ case:

(i)

$$\frac{CE(S^{RR}, \delta_S^{BSM}, \delta_I^{BSM})}{CE(S^{RR}, \delta_S^{RR}, \delta_I^{RR})}.$$

Here, the denominator represents the CE of an investor who believes the stock price follows the RR model and the stock price does follow the RR model. The numerator is the CE of the investor, who believes the stock price follows the BSM model, whereas the stock follows the RR model. Thus, this ratio measures the relative gain/loss in CE of a BSM investor in case he is wrong about the model. We study the welfare gain/loss of an investor by computing this ratio in the RR-model, and in the BSMCT model:

(ii)

$$\frac{CE(S^{BSM}, \delta_S^{RR}, \delta_I^{RR})}{CE(S^{BSM}, \delta_S^{BSM}, \delta_I^{BSM})}.$$

In this case, in the denominator we have the CE of an investor who believes the stock follows the BSM model and the stock does follow the BSM model. In the numerator we have the CE of the investor who believes that the stock price follows the RR model, whereas, in reality, the stock price follows the BSM model. Thus, the above ratio measures the relative gain/loss in CE of an RR investor in case he is wrong about the model. We do a similar exercise replacing RR model with BSMCT model.

As in previous sections, we normalize the parameters so that the means and the variances are the same at $S_0 = 1$ for the models under consideration. We see from Table 4 that if the non-BSM investor is wrong, the relative loss in CE gets larger as we move away from $S_0 = 1$, and it is far bigger for values of S that are higher than one. In other words, if the BSM model is correct, the non-BSM investor will bear a loss for using a wrong model, especially if the stock price is high and he believes it cannot go much higher, so that it is optimal for him to take a short position in the stock. The losses on the low end of the stock price are also not insignificant – when the stock price is at about 75% of the starting value of \$1.00, the CE of the non-BSM investor can take values from a quarter of the BSM investor’s gain, to a loss of multiple times that as the BSM investor’s gain, depending on the model.

On the other hand, CE losses for the BSM investor who is wrong about the model are less extreme for most values of the stock prices, and don’t vary as dramatically with those values. Nevertheless, they can be very large, and sometimes of the same order as the relative CE losses of the non-BSM investor when he is wrong. In particular, this happens in a range of moderately high values of S .

In terms of the absolute size, the relative loss is again the largest for very high values of the stock, which does not have much room to move up, and it is optimal to sell it short, which the investor who incorrectly believes in the BSM model does not do.

Overall, the most important message of this section is that in most cases the non-standard investor is exposed to a relatively higher mean-variance loss if he is wrong about the model than the loss of the standard investor when he is wrong, so the non-standard investor should make sure he indeed has superior information.

5 Conclusions and Extensions

From the standpoint of active investors, standard asset pricing models do not provide a full picture of price dynamics. As Warren Buffett notes, the very reason for being an active investor is the belief that certain securities are not valued appropriately. If that is the case, active investors cannot rely on the risk and expected return results derived from standard models. This leads to the question of how an active investor should assess the risk and expected return of an individual security. There is no general answer to that question. It depends on the type of misvaluation that the active investor believes exists. In this paper, we analyze models which account for mispricing in a reduced form. In particular, we assume that, perhaps via the application of fundamental analysis, an active investor can estimate bounds on the stock price. In this context, the risk, expected return, and the optimal investment policy of the active investor depend on the current price of the stock relative to the bounds. We consider two types of models, one with reversion from the boundaries, and another in which the investor, in addition to the boundaries, has a belief regarding the risk premium of the stock. We derive optimal investment strategies for both classes of models, when the active investor chooses between investment in a single stock, a market index and a risk-free asset. Our results show how an investor who places fundamental boundaries on a stock price will alter his investment strategy. We also demonstrate how the risk and expected return perceived by such an active investor depend on the current stock price, in distinction from standard models, in which they are independent of the price level.

One extension of our analysis would be to consider multiple stocks. More precisely, in the case of d stocks, we can replace σB in stock i with

$$\sum_{j=1}^d \sigma_{ij} B_T^{(j)}, \quad i = 1, \dots, d \quad (5.1)$$

for a given d -dimensional Brownian Motion $B = (B^{(1)}, \dots, B^{(d)})$. Appropriately choosing σ_{ij} 's produces various correlation structures on the stocks. We can also replace σ^2 for stock i with

$$\Sigma_i^2 = \sum_{j=1}^d \sigma_{ij}^2$$

and since we can write

$$\sum_{j=1}^d \sigma_{ij} B_T^{(j)} = \Sigma_i W_T^{(i)}$$

for some one-dimensional Brownian Motion $W^{(i)}$, we can use the RD model derived for individual stocks. To model a multi-dimensional distribution of the vector (S_T^1, \dots, S_T^d) , a copula approach could be employed. From the theoretical standpoint, it would also be of interest to analyze full equilibrium models in the presence of investors with RR or RD beliefs. In addition, it would be more realistic to assume that the investor does not have full information on the price bounds, and that he updates their values in a Bayesian fashion. This would require specifying the factors and the estimation procedure used for determining the bounds.

References

- [1] Basak, S. and B. Croitoru, 2000, Equilibrium Mispricing in a Capital Market with Portfolio Constraints, *The Review of Financial Studies*, 13(3), 715–748.
- [2] Black, F. and R. Litterman, 1990, Asset Allocation: Combining Investor Views With Market Equilibrium, *Goldman, Sachs & Co., Fixed Income Research*.
- [3] Black, F. and R. Litterman, 1992, Global Portfolio Optimization, *Financial Analysts Journal*, 28 – 43.
- [4] Black, F. and M. Scholes, 1973, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy*, 81:3, 637–654.
- [5] Campbell, J.Y., and R.J. Shiller, 2001, Valuation Ratios and the Long-Run Stock Market Outlook: An Update. *NBER Working Paper 8221*.
- [6] Carhart, M.M, 1997, On persistence in mutual fund performance, *Journal of Finance*, 52, 57–82.
- [7] Coval, J. D., and T.J. Moskowitz, 1999, Home bias at home: Local equity preference in domestic portfolios, *Journal of Finance*, 54, 2045-2074.
- [8] Coval, J.D., and T. J. Moskowitz, 2001, The geography of investment: Informed trading and asset prices, *Journal of Political Economy*, 109(4), 811-841.
- [9] Cvitanić, J, Lazrak, A., Martellini, L. & F. Zapatero, 2006, Dynamic Portfolio Choice with Parameter Uncertainty and the Economic Value of Analysts' Recommendations. *The Review of Financial Studies*, 19, 1113-1156.
- [10] De Long, J. B., A. Shleifer, L. Summers, and R.J. Waldmann, 1990, Positive feedback investment strategies and destabilizing rational speculation, *Journal of Finance* 45, 375-395.
- [11] Fama, E. F., 1970, Efficient Capital Markets: A Review of Theory and Empirical Work. *Journal of Finance*, 25, 383-417.
- [12] Fama, E. F., 1998, Market Efficiency, Long-Term Returns and Behavioral Finance. *Journal of Financial Economics*, 49, 283-306.
- [13] Fama, E.F. and K.R. French, 1992, The cross section of expected stock returns, *Journal of Finance*, 47, 427–465.
- [14] Fama, E.F. and K.R. French, 1993, Common risk factors in returns on stocks and bonds, *Journal of Financial Economics*, 33, 3–56.
- [15] Fama, E.F. and K.R. French, 1993, Disagreements, tastes and asset prices, *Journal of Financial Economics*, 83 667–690.
- [16] Fama, Eugene F., 1996, Multifactor explanations of asset pricing anomalies, *Journal of Finance*, 51, 55-84.

- [17] Friedman, M., 1953, The case for flexible exchange rates, in *Essays in Positive Economics*. *University of Chicago Press, Chicago*.
- [18] Lintner, J., 1965a, The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets, *Review of Economics and Statistics*, 47 (February): 13–37.
- [19] Lintner, J., 1965b, Security Prices, Risk and Maximal Gains From Diversification, *Journal of Finance*, 20 (December): 587 – 615.
- [20] Liu, J., and F. A. Longstaff 2004, Losing Money on Arbitrage: Optimal Dynamic Portfolio Choice in Markets with Arbitrage Opportunities, *The Review of Financial Studies*, 17, 611–641.
- [21] Markowitz, H. M., 1952, Portfolio Selection, *Journal of Finance* 7 (1): 77–91.
- [22] Merton, R., 1969, Lifetime portfolio selection under uncertainty: the continuous time case. *The Review of Economics and Statistics*, 51, 247–257.
- [23] Merton, R., 1971, Optimal consumption and portfolio rules in a continuous time model, *Journal of Economic Theory*, 3, 373–413.
- [24] Merton, R. C., 1973, Theory of Rational Option Pricing, *Bell Journal of Economics and Management Science*, Vol. 4, No. 1, 141–183.
- [25] Mossin, J., 1966, Equilibrium in a Capital Asset Market, *Econometrica*, 35, 768–783.
- [26] Penman, S. H., 2007, *Financial Statement Analysis and Security Valuation*, 3rd edition, McGraw-Hill, New York.
- [27] Sharpe, W.L., 1964, Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk, *Journal of Finance*, 19 (September): 425 – 442.
- [28] Sharpe, W.L., 1970, *Portfolio Theory and Capital Markets*, New York: McGraw-Hill.
- [29] Shiller, R., 1984, Stock prices and social dynamics, *Brookings Papers on Economic Activity* 2, 457-498.
- [30] Summers, L. H., 1986, Does the Stock Market Rationally Reflect Fundamental Value?, *J. of Finance*, 41, 591–600.

Table 1

Optimal allocation in the Range Reversion (RR) case

The column π^S measures the optimal proportion of wealth to be invested in the risky security, π^I measures the optimal proportion of wealth to be invested in the index, in the Range Reversion Model (RR). The column $\hat{\pi}^S$ measures the optimal proportion of wealth to be invested in the risky security, $\hat{\pi}^I$ measures the optimal proportion of wealth to be invested in the index, in the Black-Scholes Model (BSM). S_0 is the initial price of the security, the range reversion parameters are taken to be $n_L = n_U = 1$. We take $T = 1, x = 1, \mu = 0.07, \sigma = 0.21, k = 2, r = 0.01, \mu_I = 0.06, \sigma_I = 0.17, \rho = 0.2, \gamma = 1$.

PANEL A - Markowitz					PANEL B - Dynamic				
S_0	π^S	π^I	$\hat{\pi}^S$	$\hat{\pi}^I$	S_0	π^S	π^I	$\hat{\pi}^S$	$\hat{\pi}^I$
0.705	1.580	0.927	0.897	1.167	0.705	1.955	1.247	1.146	1.449
0.751	1.272	1.054	0.897	1.167	0.751	1.588	1.338	1.146	1.449
0.797	1.096	1.122	0.897	1.167	0.797	1.361	1.394	1.146	1.449
0.844	0.939	1.146	0.897	1.167	0.844	1.236	1.425	1.146	1.449
0.890	0.947	1.165	0.897	1.167	0.890	1.174	1.440	1.146	1.449
0.937	0.920	1.171	0.897	1.167	0.937	1.145	1.447	1.146	1.449
0.983	0.911	1.177	0.897	1.167	0.983	1.133	1.450	1.146	1.449
1.000	0.909	1.180	0.897	1.167	1.000	1.130	1.451	1.146	1.449
1.029	0.908	1.182	0.897	1.167	1.029	1.127	1.452	1.146	1.449
1.076	0.907	1.185	0.897	1.167	1.076	1.122	1.453	1.146	1.449
1.122	0.903	1.182	0.897	1.167	1.122	1.115	1.455	1.146	1.449
1.169	0.908	1.190	0.897	1.167	1.169	1.102	1.458	1.146	1.449
1.215	0.904	1.192	0.897	1.167	1.215	1.079	1.463	1.146	1.449
1.261	0.894	1.202	0.897	1.167	1.261	1.042	1.473	1.146	1.449
1.308	0.868	1.210	0.897	1.167	1.308	0.985	1.487	1.146	1.449
1.354	0.836	1.233	0.897	1.167	1.354	0.901	1.508	1.146	1.449
1.400	0.780	1.260	0.897	1.167	1.400	0.789	1.535	1.146	1.449
1.447	0.681	1.293	0.897	1.167	1.447	0.643	1.571	1.146	1.449
1.493	0.553	1.339	0.897	1.167	1.493	0.470	1.614	1.146	1.449
1.540	0.339	1.392	0.897	1.167	1.540	0.263	1.665	1.146	1.449
1.586	0.054	1.453	0.897	1.167	1.586	0.037	1.721	1.146	1.449
1.632	-0.315	1.515	0.897	1.167	1.632	-0.205	1.781	1.146	1.449

Table 2

Expected Returns and Variances of Returns for RR and BSMCT models.

The columns in Panel A represent the Expected Returns and in Panel B they represent the Variances of Returns corresponding to the two models. We take $T = 1, x = 1, \mu = 0.07, \sigma = 0.21, k = 2, r = 0.01, \mu_I = 0.06, \sigma_I = 0.17, \rho = 0.2, n_L = n_U = 1, \gamma = 1$.

PANEL A-Expected Returns			PANEL B-Variances		
S_0	RR	BSMCT	S_0	RR	BSMCT
0.705	1.114	1.216	0.705	0.043	0.031
0.751	1.108	1.178	0.751	0.046	0.034
0.797	1.103	1.148	0.797	0.048	0.036
0.844	1.096	1.125	0.844	0.050	0.038
0.890	1.085	1.107	0.890	0.051	0.040
0.937	1.078	1.092	0.937	0.051	0.041
0.983	1.071	1.073	0.983	0.051	0.040
1.000	1.069	1.069	1.000	0.051	0.039
1.029	1.068	1.059	1.029	0.051	0.038
1.076	1.066	1.048	1.076	0.050	0.035
1.122	1.063	1.036	1.122	0.050	0.032
1.169	1.059	1.024	1.169	0.049	0.030
1.215	1.052	1.010	1.215	0.047	0.027
1.261	1.047	0.996	1.261	0.046	0.024
1.308	1.041	0.981	1.308	0.044	0.021
1.354	1.033	0.965	1.354	0.042	0.019
1.400	1.024	0.949	1.400	0.039	0.017
1.447	1.013	0.932	1.447	0.037	0.015
1.493	1.000	0.915	1.493	0.034	0.013
1.540	0.983	0.899	1.540	0.032	0.012
1.586	0.961	0.882	1.586	0.030	0.010
1.632	0.933	0.865	1.632	0.028	0.009

Table 3

Markowitz Portfolios in the BSMCT model.

The columns represent investment weights on the stock and on the index in the case of BSMCT and BSM models. We use $T = 1, x = 1, \mu = 0.07, \sigma = 0.21, k = 2, r = 0.01, \mu_I = 0.06, \sigma_I = 0.17, \rho = 0.2$.

Markowitz portfolios				
S_0	π^S	π^I	$\hat{\pi}^S$	$\hat{\pi}^I$
0.705	2.808	0.150	1.072	0.947
0.751	2.659	0.341	1.072	0.947
0.797	2.383	0.540	1.072	0.947
0.844	2.065	0.722	1.072	0.947
0.890	1.769	0.865	1.072	0.947
0.937	1.515	0.980	1.072	0.947
0.983	1.180	1.120	1.072	0.947
1.000	1.122	1.144	1.072	0.947
1.029	0.939	1.215	1.072	0.947
1.076	0.735	1.285	1.072	0.947
1.122	0.474	1.360	1.072	0.947
1.169	0.130	1.439	1.072	0.947
1.215	-0.324	1.517	1.072	0.947
1.261	-0.907	1.585	1.072	0.947
1.308	-1.600	1.629	1.072	0.947
1.354	-2.362	1.633	1.072	0.947
1.400	-3.113	1.588	1.072	0.947
1.447	-3.763	1.494	1.072	0.947
1.493	-4.254	1.361	1.072	0.947
1.540	-4.559	1.212	1.072	0.947
1.586	-4.701	1.060	1.072	0.947
1.632	-4.714	0.918	1.072	0.947

Table 4

The Certainty Equivalents (CE) for RR and BSMCT models.

The columns represent the Certainty Equivalents for all models in both Markowitz cases: RR model and BSMCT model. We take $T = 1, x = 1, \mu = 0.07, \sigma = 0.21, k = 2, r = 0.01, \mu_I = 0.06, \sigma_I = 0.17, \rho = 0.2, n_L = n_U = 1, \gamma = 1$. In cases (i) we compute the following Certainty Equivalence ratios: $\frac{CE(S^{RR}, \delta_S^{BSM}, \delta_I^{BSM})}{CE(S^{RR}, \delta_S^{RR}, \delta_I^{RR})}$, and in cases (ii) we compute the following Certainty Equivalence ratios: $\frac{CE(S^{BSM}, \delta_S^{RR}, \delta_I^{RR})}{CE(S^{BSM}, \delta_S^{BSM}, \delta_I^{BSM})}$.

S_0	RR-i	RR-ii	BSMCT-i	BSMCT-ii
0.705	0.659	0.009	0.545	-1.689
0.751	0.784	0.241	0.588	-1.017
0.797	0.845	0.648	0.623	-0.656
0.844	0.901	0.901	0.669	-0.215
0.890	0.957	0.921	0.781	0.128
0.937	0.998	0.970	0.906	0.610
0.983	1.000	1.000	0.967	0.921
1.000	1.000	1.000	0.983	0.952
1.029	0.999	0.999	0.981	0.903
1.076	0.999	0.998	0.921	0.689
1.122	0.997	0.997	0.622	0.548
1.169	0.994	0.995	0.319	0.342
1.215	0.981	0.961	0.019	0.125
1.261	0.952	0.802	-0.116	-0.106
1.308	0.906	0.651	-0.186	-0.368
1.354	0.874	0.509	-0.211	-0.569
1.400	0.655	0.255	-0.254	-0.894
1.447	0.485	0.108	-0.294	-1.256
1.493	0.364	-0.007	-0.315	-2.125
1.540	0.189	-0.214	-0.359	-6.122
1.586	0.025	-0.368	-0.446	-8.002
1.632	-0.199	-1.125	-0.482	-9.657

Figure 1

Risk premia for the BSMCT model. We use the following parameters:

$T = 1, x = 1, \mu = 0.07, \sigma = 0.21, k = 2, r = 0.01, \mu_I = 0.06, \sigma_I = 0.17, \rho = 0.2, n_L = n_U = 1, \gamma = 1.$

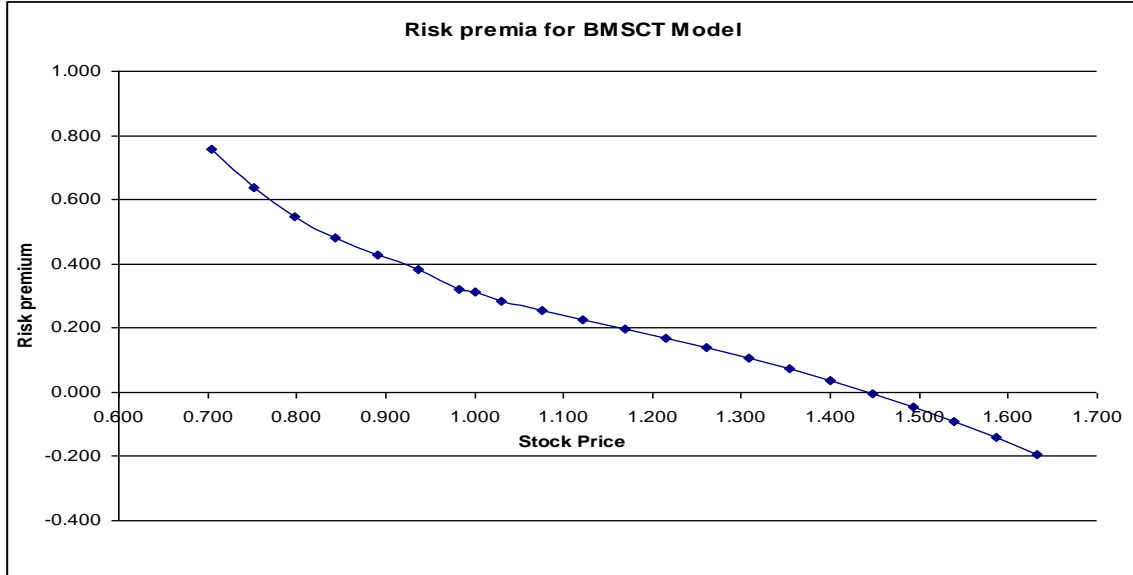


Figure 2

Betas for the RR and BSMCT models. We use the following parameters:

$T = 1, x = 1, \mu = 0.07, \sigma = 0.21, k = 2, r = 0.01, \mu_I = 0.06, \sigma_I = 0.17, \rho = 0.2, n_L = n_U = 1, \gamma = 1.$

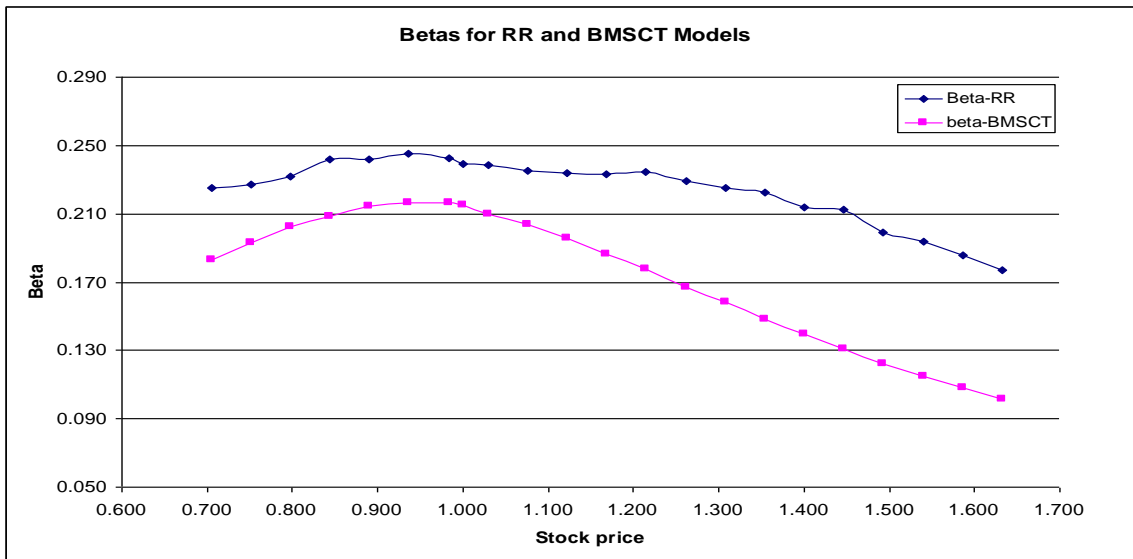


Figure 3

Instantaneous Betas for the RR and BSMCT models. We use the following parameters:

$T = 1, x = 1, \mu = 0.07, \sigma = 0.21, k = 2, r = 0.01, \mu_I = 0.06, \sigma_I = 0.17, \rho = 0.2, n_L = n_U = 1, \gamma = 1.$

