VIRTUAL DETERMINACY IN OVERLAPPING GENERATIONS MODELS

JONATHAN L. BURKE
Pepperdine University, Malibu, CA 90263-4237, U.S.A.
VIRTUAL DETERMINACY IN OVERLAPPING GENERATIONS MODELS

BY JONATHAN L. BURKE

We reappraise the significance and robustness of indeterminacy in overlapping-generations models. In any of Gale’s example economies with an equilibrium that is not locally unique, for instance, perturbing the economy by judiciously splitting each of Gale’s goods into two close substitutes restricts that indeterminacy to each period’s allocation of consumption between those substitutes. In particular, prices, interest rates, the commodity value of nominal savings (including money), and utility levels become determinate. Any indeterminacy of equilibrium consumption in the perturbed economy is thus insignificant to consumers, and some forecasting and comparative-statics policy exercises become possible.

KEYWORDS: Indeterminacy, robustness, overlapping generations, forecasting, comparative statics.

1. INTRODUCTION

THE GENERAL-EQUILIBRIUM LITERATURE motivates an appraisal of determinacy in rational-expectations models: If an equilibrium is determinate, then forecasting and comparative-statics exercises are conclusive, and rational expectations follow from underlying assumptions of perfect rationality and perfect knowledge of the environment. Debreu (1972, Section 2) interpreted any general-equilibrium model with at least one locally unique equilibrium allocation as determinate insofar as only small changes from that allocation are considered. Kehoe and Levine (1985) described a stronger standard for determinacy in stationary overlapping-generations models. They supposed exogenous, past consumption achieved a steady state and required for determinacy that the particular equilibrium replicating past steady-state consumption from period 1 onward be locally unique among all (stationary and nonstationary) equilibria. One interpretation of the latter standard is consumers have only local information about other consumers’ preferences near past consumption, and so only small changes from past consumption are considered.

For simple indeterminacy examples, consider stationary, pure-exchange overlapping-generations models with one consumer per generation, two periods per lifetime, and one (aggregate) good available per period. Under standard conditions, Gale (1973) showed almost every such model has two steady states (autarky and the golden rule), and one steady state is locally unique and

1Preliminary versions of this paper were presented at the North American Summer Meetings of the Econometric Society at UCLA, and at the first annual CARESS–Cowles conference on general equilibrium and its applications at Yale University. Thoughtful critiques by seminar participants and university colleagues provoked a significant revision. My thanks to one especially thorough referee of this journal for his clarification of my central proof and to Levon Goukasian for help polishing the final draft.
the other is not. 2 For the latter steady state, there is a continuum of nearby nonstationary equilibrium allocations. Under Kehoe and Levine’s standard, the model is thus determinate if exogenous, past consumption were the locally unique steady state, but indeterminate if past consumption were the other state. The literature accepts such indeterminacy as robust since it remains in one-good models after any suitably small perturbation of preferences. However, indeterminacy is no longer robust after making two changes to standard analysis.

First, allow perturbations of preferences to disaggregate individual goods into substitutes. Specifically, consider a one-good-per-period model to be equivalent to a disaggregated version with two perfect substitutes (say, early and late consumption) available per period, and consider a one-good model to be perturbed by any sequence of two-good models that converge to the two-good disaggregated version of the one-good model.

Second, weaken Kehoe and Levine’s standard for determinacy to require that the particular equilibrium replicating past steady-state consumption be virtually locally unique, rather than locally unique in the standard sense. “Virtual” local uniqueness allows there to be a continuum of nonstationary equilibrium allocations in any neighborhood of the particular equilibrium, but prices, rates of return, savings values, and utility levels are constant across all those equilibria when the neighborhood is sufficiently small. (For example, the first consumer born in the economy achieves the same utility level in each equilibrium, as does the second consumer and so on.) In particular, any indeterminacy of equilibrium consumption in the neighborhood is thus insignificant to consumers.

Here is the plan of the rest of the paper. Sections 2 and 3 begin precise analysis with a definition of economies and equilibria with two goods available per period and discrete time, starting in period 1. The definition of economies is standard, and includes all the two-good economies we need to perturb away the significant indeterminacy in Gale’s one-good models. Following Kehoe and Levine (1985), equilibria are defined and interpreted as the possible outcomes of a particular comparative-statics exercise. 3

Sections 4–6 define virtual local uniqueness and contain the main results (Theorem 5.1 and Theorem 5.2), which prove a representative example of

---

2 In particular, having every steady state locally unique is exceptional; it only happens when there is a unique equilibrium. Overlapping-generations analysis thus differs from Debreu’s (1972, Section 2) analysis for a finite number of consumers and goods, where the sufficient conditions found for guaranteeing at least one equilibrium is locally unique are the same as the conditions guaranteeing every equilibrium is locally unique.

3 Specifically, suppose exogenous, past consumption achieved a steady state. Then an unexpected sunspot shock occurs (which does not change endowments or preferences), and consumers may revise their expectations and so follow a new perfect-foresight equilibrium path from period 1 onward.
Gale’s indeterminacy models can be perturbed so that the particular equilibrium replicating past consumption becomes virtually locally unique. Thus significant indeterminacy is not robust. Finally, Section 7 considers further implications of Theorem 5.1 and Theorem 5.2, and discusses known extensions.

2. ECONOMIES

This section defines a typical universe of pure-exchange overlapping-generations economies with endowments and preferences that are stationary across generations. There are a countable number of time periods, indexed \( t \in \mathbb{N} := \{1, 2, \ldots\} \), two nonstorable goods available per period, indexed \( i = 1, 2 \), and a countable number of consumers. Consumer 0 lives in period 1 (his old age), and consumer \( t \in \mathbb{N} \) in periods \( t \) (youth) and \( t + 1 \) (old age).

Throughout the paper, fix a positive endowment \( e = (a_1, a_2, b_1, b_2) \) for consumer \( t \in \mathbb{N} \) of the two goods in youth, \( a_i > 0 \), and the two goods in old age, \( b_i > 0 \). (The endowment is the same for every consumer \( t \in \mathbb{N} \).) Consumer 0 receives the old-age endowment, \( (b_1, b_2) \).

Let \( x_{t}^{i,t} \) denote typical consumption of good \( i \) in period \( t \in \mathbb{N} \) by consumer \( \tau = t - 1, t \) in our first definition:

**DEFINITION 1:** A consumption vector \( x_0 = (x_0^{1,1}, x_0^{1,2}) \in \mathbb{R}^2_+ \) for consumer 0 denotes the old-age (period 1) consumption \( x_0^{1,i} \) of each good \( i \), and a consumption vector \( x_t = (x_t^{1,1}, x_t^{1,2}, x_{t+1}^{1,1}, x_{t+1}^{1,2}) \in \mathbb{R}^4_+ \) for consumer \( t \in \mathbb{N} \) denotes consumption \( x_t^{i,t} \) in youth and \( x_{t+1}^{i,t} \) in old age.

Parameterize an economy by a utility function \( u(x_t) = u(x_t^{1,1}, x_t^{1,2}, x_{t+1}^{1,1}, x_{t+1}^{1,2}) \) representing preferences for consumer \( t \in \mathbb{N} \) over consumption vectors (Definition 1). (The utility function is the same for every consumer \( t \in \mathbb{N} \).) The definition of equilibrium (Definition 3) in Section 3 represents consumer 0’s preferences over old-age consumption \( x_0 = (x_0^{1,1}, x_0^{1,2}) \) by the same utility function \( u(x_0^{0,1}, x_0^{0,2}, x_0^{1,1}, x_0^{1,2}) \) above, after restricting young-age consumption \((x_0^{0,1}, x_0^{0,2})\) to a particular steady state.

The universe of economies are those utility functions \( u : \mathbb{R}^4_+ \rightarrow [-\infty, +\infty) \) satisfying these assumptions:

**ASSUMPTION 1:** The function \( u \) is concave and nondecreasing over \( \mathbb{R}^4_+ \).

**ASSUMPTION 2:** The function \( u \) is finite-valued (\( u > -\infty \)) and increasing over those vectors at least as good as the endowment, \( u(x_t^{1,1}, x_t^{1,2}, x_{t+1}^{1,1}, x_{t+1}^{1,2}) \geq u(e) \).

For example, the modified sum-of-logs function \( \ln(x_t^{1,1} + x_t^{1,2}) + \ln(x_{t+1}^{1,1} + x_{t+1}^{1,2}) \) satisfies both assumptions.
3. EQUILIBRIA

This section defines a typical class of stationary and nonstationary equilibria for each economy \( u \) satisfying Assumptions 1 and 2.

**DEFINITION 2:** An allocation is a path \( x = (x_t)_{t \in \{0\} \cup \mathbb{N}} \) of consumption vectors, with \( x_0 = (x_{0,1}^1, x_{0,1}^2) \in \mathbb{R}_+^2 \) and \( x_t = (x_{t,1}^1, x_{t,2}^2, x_{t+1,1}^1, x_{t+1,2}^2) \in \mathbb{R}_+^4 \) (Definition 1), that balances materials\(^4\)

\[
(1) \quad x_{t-1}^{1,i} + x_t^{1,i} = b^i + a^i \quad (t \in \mathbb{N}, \ i = 1, 2).
\]

In the following definition of equilibrium, good 1 is numéraire, \( p^1 > 0 \) is the relative price of good 2, \( R^t > 0 \) is the gross rate of return on savings from period \( t \) to period \( t + 1 \) (principal plus interest in terms of the numéraire), and \( M \) is the period 1 numéraire value of consumer 0’s nominal savings.

**DEFINITION 3:** For any young-age consumption \( (\tilde{x}^{0,1}, \tilde{x}^{0,2}) \in \mathbb{R}_+^2 \) by consumer 0, an allocation \( x = (x_t)_{t \in \{0\} \cup \mathbb{N}} \) is an equilibrium (or equilibrium continuation of \( (\tilde{x}^{0,1}, \tilde{x}^{0,2}) \)) if it is supported by some price sequence \( p = (p_t)_{t \in \mathbb{N}} \in \mathbb{R}_+^N \), rate-of-return sequence \( R = (R^t)_{t \in \mathbb{N}} \in \mathbb{R}_{++}^N \), and savings value \( M \in \mathbb{R} \). Specifically, old-age consumption \( x_0 = (x_{0,1}^1, x_{0,1}^2) \in \mathbb{R}_+^2 \) by consumer 0 maximizes utility \( u(x^{0,1}, x^{0,2}, x_0^1, x_0^2) \) over all vectors \( x_0 \in \mathbb{R}_+^2 \) satisfying the budget constraint

\[
(2) \quad (x_0^{1,1} - b^1) + p^1(x_0^{1,2} - b^2) = M
\]

and lifetime consumption \( x_t = (x_t^{1,1}, x_t^{1,2}, x_t^{1+1,1}, x_t^{1+1,2}) \) by consumer \( t \in \mathbb{N} \) maximizes utility \( u(x_t) \) over all vectors \( x_t \in \mathbb{R}_+^4 \) satisfying

\[
(3) \quad (x_t^{1,1} - a^1) + p^t(x_t^{1,2} - a^2) + \frac{(x_t^{1+1,1} - b^1) + p^{t+1}(x_t^{1+1,2} - b^2)}{R^t} = 0.
\]

**DEFINITION 4:** An equilibrium continuation \( x = (x_t)_{t \in \{0\} \cup \mathbb{N}} \) of (young-age) consumption \( (\tilde{x}^{0,1}, \tilde{x}^{0,2}) \in \mathbb{R}_+^2 \) is a stationary equilibrium if consumption is stationary across consumers. That is, writing (old-age) consumption \( x_0 = (\bar{x}^{1,1}, \bar{x}^{1,2}) \) for consumer 0, then \( x_t = (\tilde{x}^{0,1}, \tilde{x}^{0,2}, \bar{x}^{1,1}, \bar{x}^{1,2}) \) for consumer \( t \in \mathbb{N} \).

**DEFINITION 5:** A steady state is the consumption vector \( \bar{x} = (\tilde{x}^{0,1}, \tilde{x}^{0,2}, \bar{x}^{1,1}, \bar{x}^{1,2}) \in \mathbb{R}_+^4 \) received by every consumer in a stationary equilibrium.

Since utility is assumed increasing over those vectors at least as good as the endowment (Assumption 2), the set of equilibria would be unchanged

\(^4\)Section 2 fixed individual consumer endowments \( e = (a^1, a^2, b_1^1, b_1^2) \) throughout the paper.
if prices $p'$ were allowed to be zero or negative. Likewise, the set of equilibria would be unchanged if budget constraints were defined by a sequence $(q^{t,1}, q^{t,2})$, $(q^{t,1}, q^{t,2}), \ldots$ of present-value prices. For example, the budget constraint (3) of consumer $t \in \mathbb{N}$ evidently has the same feasible solutions $x_t = (x_t^{t,1}, x_t^{t,2}, x_t^{t+1,1}, x_t^{t+1,2})$ as the alternative constraint

$$q^{t,1}(x_t^{t,1} - a^1) + q^{t,2}(x_t^{t,2} - a^2) + q^{t+1,1}(x_t^{t+1,1} - b^1) + q^{t+1,2}(x_t^{t+1,2} - b^2) = 0$$

defined by positive present-value prices.\(^6\)

Kehoe and Levine (1985) interpreted the set of all equilibrium continuations of a fixed steady state (Definitions 3 and 5) as the possible outcomes of a particular comparative-statics exercise. They supposed exogenous, past consumption before period 1 achieved a steady state $\tilde{x} = (\tilde{x}^{0,1}, \tilde{x}^{0,2}, \tilde{x}^{1,1}, \tilde{x}^{1,2})$, and then an unexpected shock occurs. In the current setting, just consider an unexpected sunspot, which does not change endowments or preferences. After the shock, consumers may revise their expectations and so follow any equilibrium continuation (Definition 3) of the steady-state young-age consumption $(\tilde{x}^{0,1}, \tilde{x}^{0,2})$. Specifically, consumers can buy and sell commodities on complete spot markets at perfectly forseen prices, and consumer 0 carries over nominal savings from the past.\(^6\)

4. VIRTUAL LOCAL UNIQUENESS

This section compares a typical definition of local uniqueness of equilibria to virtual local uniqueness for each economy $u$ satisfying Assumptions 1 and 2. Following Gale (1973) and Kehoe and Levine (1985), endow the ambient space $(\mathfrak{M}_2^+ \times \mathfrak{M}_4^+ \times \cdots)$ of consumption paths (Definition 2) with the uniform (or sup) topology (Munkres (1975, pp. 119, 122)).\(^7\)

**DEFINITION 6:** A stationary equilibrium $x = (x_t)_{t \in \{0\} \cup \mathbb{N}}$, with steady state $\tilde{x} = (\tilde{x}^{0,1}, \tilde{x}^{0,2}, \tilde{x}^{1,1}, \tilde{x}^{1,2})$, is **locally unique** if, for some neighborhood $X \subset (\mathfrak{M}_2^+ \times \mathfrak{M}_4^+ \times \cdots)$ of $x$, the path $x$ is the only allocation in $X$ that is an equilibrium continuation (Definition 3) of young-age consumption $(\tilde{x}^{0,1}, \tilde{x}^{0,2})$.

**DEFINITION 7:** A stationary equilibrium $x = (x_t)_{t \in \{0\} \cup \mathbb{N}}$, with steady state $\tilde{x} = (\tilde{x}^{0,1}, \tilde{x}^{0,2}, \tilde{x}^{1,1}, \tilde{x}^{1,2})$, is **virtually locally unique** if, for some neighborhood $X \subset$

---

\(^5\)Specifically, $q^{t,1} := 1 / \prod_{t=1}^{r'} R^t$ (with null product $q^{1,1} := 1$) and $q^{t,2} := p' q^{t,1}$ derive present-value prices from the $p'$ and $R^t$, while $p' := q^{t,2} / q^{t,1}$ and $R^t := q^{t,1} / q^{t+1,1}$ derive the $p'$ and $R^t$ from present-value prices.

\(^6\)Such savings is called “money” in the literature when its commodity value is nonnegative.

\(^7\)The uniform topology is generated by the metric $\sup_{t \in \{0\} \cup \mathbb{N}} \min \| x_t - \tilde{x}_t \|, 1 \}$, defined over pairs of paths $x = (x_t)_{t \in \{0\} \cup \mathbb{N}}$ and $\tilde{x} = (\tilde{x}_t)_{t \in \{0\} \cup \mathbb{N}}$. 

---
of $x$, there is a unique triple $(p, R, M)$ of a price sequence $p = (p_t)_{t \in N}$, a rate-of-return sequence $R = (R_t)_{t \in N}$, and a savings value $M$ that support ((2), (3)) every allocation in $X$ that is an equilibrium continuation (Definition 3) of young-age consumption $(\bar{x}^{0.1}, \bar{x}^{0.2})$.

Kehoe and Levine (1985) interpreted local uniqueness of a stationary equilibrium (Definition 6) by supposing exogenous, past consumption achieved a steady state $\bar{x}$ and that consumers have only local information about other consumers' preferences near $\bar{x}$. Hence, only allocations in the neighborhood $X$ are considered. Local uniqueness of the particular equilibrium $x$ replicating past steady-state consumption from period 1 onward thus implies determinacy, and forecasting and comparative-statics policy exercises are possible.

"Virtual" local uniqueness (Definition 7) has virtually the same interpretation and implications: there may be many nonstationary equilibrium allocations in the neighborhood $X$, but prices, rates of return, and savings values are constant across all those equilibria. Any nonuniqueness of consumption in the neighborhood is thus insignificant to consumers, and forecasting and comparative-statics policy exercises are still possible.

5. PERTURBING INDETERMINACY EXAMPLES

This section perturbs away significant indeterminacy for a representative example of Gale’s indeterminacy models. Gale’s models have only one good per period. To fix an example, fix the endowment at 6 for the good in youth and 2 in old age, and fix the sum-of-logs utility function

\[(4) \quad \ln(x^0) + \ln(x^1)\]

to represent preferences over youth $x^0$ and old-age $x^1$ consumption. Gale (1973) proved there are two stationary equilibria (and steady states): the golden rule and autarky. The golden rule is locally unique (and virtually locally unique), but autarky is not locally unique (and not virtually locally unique). The model is thus indeterminate, under Kehoe and Levine’s standard, if exogenous, past consumption was autarkic.

The first part of our perturbation of Gale’s example requires there be at least one way to disaggregate goods (say, into early and late moments within each period) so that there is a complete spot market for each of two goods available per period. Although we could accommodate any disaggregation and any individual consumer endowments $(a^1, a^2, b^1, b^2)$ such that $a^1 + a^2 = 6$ and $b^1 + b^2 = 2$, for the rest of the paper just consider the disaggregation for which individual consumer endowments are

\[(5) \quad e = (a^1, a^2, b^1, b^2) = (3, 3, 1, 1).\]

That is, $a^i = 3$ units of each good in youth, and $b^i = 1$ of each in old age.
One underlying implicit assumption in any general-equilibrium model with a finite number of goods available is that if any one of the goods were disaggregated into two (or more) goods, then those disaggregated goods must be perfect substitutes. Thus Gale’s model implicitly assumes his one-good-per-period function \( v(4) \) to be equivalent to this disaggregated version \( v \) with two perfect substitutes per period

\[
v(x_{0,1}, x_{0,2}, x_{1,1}, x_{1,2}) := \ln(x_{0,1} + x_{0,2}) + \ln(x_{1,1} + x_{1,2}).
\]

There is an evident equivalence of steady states and of equilibria between the one-good version \( v(4) \) and the two-good version \( v(6) \) of Gale’s example. In particular, the autarkic steady state in the one-good version is equivalent, in the two-good version, to a *barter* steady state—that is, a steady state \( \bar{x} = (\bar{x}_{0,1}, \bar{x}_{0,2}, \bar{x}_{1,1}, \bar{x}_{1,2}) \) in which savings across periods have zero value ((2), (5))

\[
(\bar{x}_{0,1} - 3) + p'(\bar{x}_{0,2} - 3) = 0, \quad (\bar{x}_{1,1} - 1) + p'(\bar{x}_{1,2} - 1) = 0 \quad (t \in \mathbb{N})
\]

for the price sequence \( (p')_{t \in \mathbb{N}} \) supporting the stationary equilibrium associated with the steady state (Definition 5).

**THEOREM 5.1:** Gale’s economy \( v(6) \) has an infinite number of barter steady states, indexed

\[
x(\lambda) := \lambda(4, 2, 0, 2) + (1 - \lambda)(2, 4, 2, 0), \quad \text{for} \quad \lambda \in [0, 1].
\]

The stationary equilibrium associated with any one of those barter steady states is not locally unique and not virtually locally unique.9

Theorem 5.1 is proved in the Appendix. Theorem 5.1 implies that the significant indeterminacy in Gale’s one-good model \( v(4) \) remains in the two-good version \( v(6) \) if exogenous, past consumption were one of the barter steady states (8).

The remainder of our perturbation is to consider Gale’s one-good example \( v(4) \) to be perturbed by a particular sequence of two-good economies \( u^k \) that converge to the two-good disaggregated version \( v(6) \) of the one-good example.

---

8 That assumption is commonly recognized in applied general-equilibrium models. If one only has enough data to define “hats” as a good, then one must assume “black hats” and “white hats” are perfect substitutes.

9 There are also an infinite number of nonbarter steady states, but the stationary equilibrium associated with any one of those is virtually locally unique and so is not central to our analysis.
THEOREM 5.2: There exists a concave and increasing $C^\infty$ function $u : \mathbb{R}_+^4 \to \mathbb{R}$ such that, for $k = 1, 2, \ldots$, the perturbed economy\(^{10}\)

(9) \quad \begin{align*}
u^k := v + \frac{1}{k} u
\end{align*}

has the same set of barter (8) steady states $x(\lambda)$, $\lambda \in [0, 1]$, as the Gale economy $v$. But the stationary equilibrium associated with any one of those barter steady states $x(\lambda)$ with $\lambda > 0$ is now virtually locally unique for the perturbed economy (9).

Likewise, there exists another perturbation function $u$ satisfying all the properties above, but the stationary equilibrium associated with any barter steady state $x(\lambda)$ with $\lambda < 1$ is now virtually locally unique for the perturbed economy.

Theorem 5.2 shows each perturbed economy overturns the significant indeterminacy (lack of virtual local uniqueness) Theorem 5.1 finds in the Gale economy $v$ (6) if exogenous, past consumption were one of the barter steady states $x(\lambda)$. Additionally, because the perturbation function $u$ is $C^\infty$, the sequence of perturbed economies (9) evidently converges to the Gale economy $v$ for any of the topologies typically considered in the literature.

6. PROOF OF THEOREM 5.2

By symmetry, it is sufficient to prove the first paragraph of Theorem 5.2.

STEP 1: Construct the perturbation function $u$. Consider the multivariate polynomial

(10) \quad \phi := \frac{1}{6}(x^{0,1} + x^{0,2}) + \frac{1}{2}(x^{1,1} + x^{1,2}) - \varepsilon(x^{0,1} + x^{0,2} - 6)^2(108 + (x^{1,2} + 1)^2)

over all vectors $x = (x^{0,1}, x^{0,2}, x^{1,1}, x^{1,2}) \in \mathbb{R}_+^4$. First-order partial derivatives

(11) \quad \begin{align*}
\phi_1 &= \phi_2 = \frac{1}{6} - 2\varepsilon(x^{0,1} + x^{0,2} - 6)(108 + (x^{1,2} + 1)^2), \\
\phi_3 &= \frac{1}{2}, \quad \phi_4 = \frac{1}{2} - 2\varepsilon(x^{0,1} + x^{0,2} - 6)^2(x^{1,2} + 1)
\end{align*}

and second-order derivatives prove that if parameter $\varepsilon$ is chosen to be positive and sufficiently small, then $\phi$ is concave and increasing over all vectors $(x^{0,1}, x^{0,2}, x^{1,1}, x^{1,2}) \leq (5, 5, 5, 5)$. Fix such a parameter $\varepsilon > 0$. Hence, there ex-

\(^{10}\)Evidently, the function $u^k := v + \frac{1}{k} u$ satisfies Assumptions 1 and 2 for economies because the Gale economy $v$ (6) satisfies those assumptions, and the function $u$ is everywhere finite-valued, concave, and increasing.
ists a concave and increasing $C^\infty$ function $u : \mathbb{R}_+^4 \to \mathbb{R}$ that equals $\phi$ over those vectors in the feasible set

\begin{equation}
F := \{(x^{0.1}, x^{0.2}, x^{1.1}, x^{1.2}) \in \mathbb{R}_+^4 \mid (x^{0.1}, x^{0.2}, x^{1.1}, x^{1.2}) \leq (4, 4, 4, 4)\}.
\end{equation}

Material balance ((1), (5)) implies the set $F$ (13) contains every steady state and every consumption for consumer $t \in \mathbb{N}$ that is part of any equilibrium.

By construction, functions $u$ and $\phi$ have the same first-order partial derivatives ((11), (12)) over $F$. Those derivatives ((11), (12)) for function $u$ evidently have the following properties for each vector $(x^{0.1}, x^{0.2}, x^{1.1}, x^{1.2}) \in F$:

**PROPERTY 1:** $u_1 = u_2$.

**PROPERTY 2:** $u_3 \geq u_4$ over $F$ with equality if and only if $x^{0.1} + x^{0.2} = 6$.

**PROPERTY 3:** $u_1 = \frac{1}{3} u_4$ if both $x^{0.1} + x^{0.2} = 6$ and $x^{1.1} + x^{1.2} = 2$. \hspace{1cm} \textit{Q.E.D.}

**STEP 2:** For $k = 1, 2, \ldots$, prove the perturbed economy $u^k = v + \frac{1}{k} u$ ((6), (9)) has the same set of barter steady states as the Gale economy $v$ (8).

**PROOF:** The perturbed economy ((6), (9))

\begin{equation}
\begin{aligned}
u^k(x^{0.1}, x^{0.2}, x^{1.1}, x^{1.2}) &= \ln(x^{0.1} + x^{0.2}) + \ln(x^{1.1} + x^{1.2}) + \frac{1}{k} u(x^{0.1}, x^{0.2}, x^{1.1}, x^{1.2})
\end{aligned}
\end{equation}

evidently inherits Properties 1, 2, and 3 from economy $u$. That is, for each vector $(x^{0.1}, x^{0.2}, x^{1.1}, x^{1.2}) \in F$:

**PROPERTY 1’:** $u^k_1 = u^k_2$.

**PROPERTY 2’:** $u^k_3 \geq u^k_4$ over $F$ with equality if and only if $x^{0.1} + x^{0.2} = 6$.

**PROPERTY 3’:** $u^k_1 = \frac{1}{3} u^k_4$ if both $x^{0.1} + x^{0.2} = 6$ and $x^{1.1} + x^{1.2} = 2$.

To compute the barter steady states of the perturbed economy $u^k$, consider consumption $x(\lambda) = \lambda(4, 2, 0, 2) + (1 - \lambda)(2, 4, 2, 0)$ for some $\lambda \in [0, 1]$. That vector satisfies the hypotheses of Properties 2’ and 3’. Hence, $u^k_1 = u^k_2 = \frac{1}{3} u^k_3 = \frac{1}{3} u^k_4$. That is, for the stationary prices $p^i = 1$ and rates of return

\textsuperscript{11}For example, set $u(x^{0.1}, x^{0.2}, x^{1.1}, x^{1.2}) := \phi(\psi(x^{0.1}), \psi(x^{0.2}), \psi(x^{1.1}), \psi(x^{1.2}))$, where $\psi : \mathbb{R}_+ \to [0, 5]$ is a concave and increasing $C^\infty$ function that equals the identity on $[0, 4]$. Such a function $\psi$ can be constructed using a convolution (Mas-Colell (1985, p. 41)).
$R^t = 1/3$, the utility gradient $\partial u^k = (u_1^k, u_2^k, u_3^k, u_4^k)$ at $x(\lambda)$ is proportional to $(1, p', 1/R^t, p^{t+1}/R^t)$, and for $p' = 1, x(\lambda)$ evidently has zero savings across periods (7). Hence, $x(\lambda)$ is a barter steady state (7) with stationary prices $p' = 1$, rates of return $R^t = 1/3$, and savings value $M = 0$. Thus, the perturbed economy $u^k$ has all of the barter steady states $x(\lambda), \lambda \in [0, 1]$, of economy $v$.

To prove the converse, consider any barter steady state $\tilde{x} = (\tilde{x}^{0,1}, \tilde{x}^{0,2}, \tilde{x}^{1,1}, \tilde{x}^{1,2})$ of the perturbed economy $u^k$, and consider any price sequence $(p')_{t \in \mathbb{N}}$ supporting the stationary equilibrium associated with the steady state $\tilde{x}$. Zero young-age savings (7)

$$(\tilde{x}^{0,1} - 3) + p'(\tilde{x}^{0,2} - 3) = 0$$

implies price $p' = 1$ since, otherwise, budget-constrained utility maximization ((3), (5)) by consumer $t \in \mathbb{N}$ and $u_1^k = u_2^k$ over the feasible set $F$ (Property 1') implies either $\tilde{x}^{0,1} > 6$ or $\tilde{x}^{0,2} > 6$, either of which contradicts $\tilde{x} \in F$ (13). But $p' = 1$ and zero savings (7) imply

$$\tilde{x}^{0,1} + \tilde{x}^{0,2} = 6,$$

which with material balance ((1), (5)) equations $\tilde{x}^{0,1} + \tilde{x}^{1,1} = 4$ and $\tilde{x}^{0,2} + \tilde{x}^{1,2} = 4$ imply

$$(\tilde{x}^{0,1}, \tilde{x}^{0,2}, \tilde{x}^{1,1}, \tilde{x}^{1,2}) = (2 + 2\lambda, 4 - 2\lambda, 2 - 2\lambda, 2\lambda)$$

for $\lambda := (\tilde{x}^{0,1} - 2)/2$. That is, the barter steady state $\tilde{x} = (\tilde{x}^{0,1}, \tilde{x}^{0,2}, \tilde{x}^{1,1}, \tilde{x}^{1,2})$ equals $x(\lambda)$ (8). Finally, the nonnegativity of consumption $\tilde{x}$ implies $\lambda \in [0, 1]$, as required.

**Step 3:** For $k = 1, 2, \ldots$, prove the stationary equilibrium associated with any one of the barter steady states $x(\lambda)$ with $\lambda > 0$ is virtually locally unique for the perturbed economy $u^k = v + \frac{1}{k} u$ (9).

**Proof:** Since $\lambda > 0$, consumption $x(\lambda)$ is positive in its first, second, and fourth components (8). Hence, define the required neighborhood\(^{12}\) $X \subset (\mathbb{R}_+^4 \times \mathbb{R}_+^4 \times \cdots)$ of the stationary equilibrium with steady state $x(\lambda)$ to be those paths $x = (x_t)_{t \in [0, \infty) \cap \mathbb{N}}$ with consumption positive in their first, second, and fourth components

$$(14) \quad x_t^{l,1} > 0, \quad x_t^{l,2} > 0, \quad \text{and} \quad x_t^{l+1,2} > 0$$

for each consumer $t \in \mathbb{N}$.

It remains to consider any equilibrium continuation ((2), (3)) $x = (x_t)_{t \in [0, \infty) \cap \mathbb{N}}$ of steady state $x(\lambda)$ that is in the neighborhood (14), and show there is a unique

\(^{12}\) The set $X$ is a neighborhood of the stationary equilibrium because the interior of $X$ contains the stationary equilibrium.
value for supporting prices $p^t$, rates of return $R^t$, and savings values $M$. Specifically, we will show $p^t = 1$, $R^t = 1/3$, and $M = 0$ for each equilibrium continuation $x = (x_t)_{t \in [0] \cup \mathbb{N}}$ in the neighborhood (14).

Lagrangian first-order conditions for budget-constrained utility maximization (3) for consumer $t \in \mathbb{N}$ with positive consumption in first, second, and fourth components (14) imply

\begin{align}
(15) \quad p^t &= \frac{u^k_t(x_t)}{u^1_t(x_t)}, \\
(16) \quad \frac{u^k_4(x_t)}{u^3_4(x_t)} &\geq p^{t+1}, \\
(17) \quad \frac{R^t}{p^{t+1}} &= \frac{u^k_t(x_t)}{u^1_t(x_t)}.
\end{align}

Hence, Property 1’ and the first equality (15) determine prices $p^t = 1$ for each period $t \in \mathbb{N}$.

Hence, for consumer $t \in \mathbb{N}$, price $p^{t+1} = 1$ and the inequality (16) imply $u^k_4(x_t) \geq u^3_4(x_t)$, which with Property 2’ implies the young-age consumption aggregation

\begin{align}
(18) \quad x^{t,1}_t + x^{t,2}_t = 6.
\end{align}

That young-age restriction (18) holding in each period combines with material balance ((1), (5)) to restrict old-age consumption

\begin{align}
(19) \quad x^{t+1,1}_t + x^{t+1,2}_t = 2.
\end{align}

Hence, the aggregation restrictions ((18), (19)) and Property 3’ imply $u^k_1(x_t) = \frac{1}{3} u^3_4(x_t)$, which with price $p^{t+1} = 1$ and the second equality (17) determines the rate of return, $R^t = 1/3$.

Finally, price $p^t = 1$ and the old-age restriction $x^{t,1}_0 + x^{t,2}_0 = 2$ (19) substitute into consumer 0’s budget constraint ((2), (5)) to determine the savings value, $M = 0$. Q.E.D.

7. CONCLUSION

Theorems 5.1 and 5.2 were specialized to perturb away the significant indeterminacy for just a representative example of Gale’s indeterminacy models. That should provoke generalizations and extensions. One known generalization is that the proof of Theorem 5.2 in Section 6 can be adapted from log-linear utility (4) with a particular disaggregation of endowments (5) to accommodate any of Gale’s indeterminacy models and any disaggregation of endowments. One known extension is that the conclusion that the equilibrium in
the perturbed economies is virtually locally unique can be reformulated and strengthened to a conclusion that the correspondence \( u \mapsto S(u) \) of economies \( u \) into the set \( S(u) \) of supporting prices, rates of return, and savings values is single-valued and continuous at the perturbed economies \( u^k \). Roughly, continuity means that significant indeterminacy returns slowly (the support set \( S(u) \) remains small) as one moves away from the perturbed economies.

To guide further generalizations and extensions, note that our perturbation of indeterminant overlapping-generations models into virtually determinate models required both of our nonconventional steps: (1) perturbing models with one good into models with two perfect or close substitutes, and (2) interpreting indeterminacy in the division of goods among substitutes as insignificant. In any model, taking one step alone is insufficient and can mislead one to think that determinate models can be perturbed into indeterminate models. For example, consider a partial-equilibrium model with a single consumption variable and determinate equation: \( f(x) = 0 \) with a single solution at \( \bar{x} \). Step 1 alone considers \( f(x) = 0 \) to be equivalent to the bivariate equation \( f(x_1 + x_2) = 0 \), and the latter equation seems to be indeterminate because \( f(x_1 + x_2) = 0 \) has an infinite number of solutions, defined by \( x_1 + x_2 = \bar{x} \). But step 2 interprets that indeterminacy in the division of \( \bar{x} \) between \( x_1 \) and \( x_2 \) to be insignificant.

Finally, both step 1 and step 2 make sense when variables measure economic consumption or production. It remains to be seen whether there are other applications.

**APPENDIX: PROOF OF THEOREM 5.1**

Step 2 in Section 6 proves any economy satisfying Properties 1 and 3 and \( u_3(x(\lambda)) = u_4(x(\lambda)) \) has the required set of barter steady states (8). The Gale economy \( v \) (6) satisfies those properties, and so it has the required barter steady states. It remains to prove the stationary equilibrium associated with any one of those barter steady states \( x(\lambda) = \lambda(4, 2, 0, 2) + (1 - \lambda)(2, 4, 2, 0) \), \( \lambda \in [0, 1] \), is not locally unique and not virtually locally unique.

For any sequence of scalers \( z \in [0, 1] \), consider the path \( x = (x_t)_{t \in [0,1]} \) of consumption vectors

\[
(20) \quad x_0 := \lambda(0, 2) + (1 - \lambda)(2, 0) + (z^0, 0) \quad \text{and} \quad x_t := x(\lambda) + (-z^{t-1}, 0, z^t, 0),
\]

where the pair \( \lambda(0, 2) + (1 - \lambda)(2, 0) \) consists of the third and fourth components of \( x(\lambda) \). The path \( x \) is evidently nonnegative and balances materials ((1), (5), and (20), and so is an allocation Definition 2).

Compute marginal utilities

\[
v_1(x_t) = v_2(x_t) = \frac{1}{6 - z^{t-1}} \quad \text{and} \quad v_3(x_t) = v_4(x_t) = \frac{1}{2 + z^t}.
\]
along the consumption path (20), and consider the price sequence and rate-of-return sequence

\[ p' := 1 \quad \text{and} \quad R' := \frac{2 + z'}{6 - z'} \]  

Since the utility gradient \( \partial v = (v_1, v_2, v_3, v_4) \) at \( x_t \) is proportional to \( (1, p', 1/R', p^{t+1}/R') \), the Lagrangian first-order conditions for budget-constrained utility maximization by consumer \( t \in \mathbb{N} \) at \( x_t \) ((3), (20)) reduce to the budget constraint

\[ -z^{t-1} + \frac{6 - z^{t-1}}{2 + z'} z' = 0. \]  

Evidently, that constraint defines the first-order difference equation \( z' = f(z^{t-1}) := z^{t-1}/(3 - z^{t-1}) \), which satisfies \( 0 = f(0) \) and \( 0 < f'(z^{t-1}) < 1 \) for \( z^{t-1} \in [0, 1] \). Therefore, for any neighborhood \( X \subset (\mathbb{R}_+^2 \times \mathbb{R}_+^4 \times \cdots) \) of the stationary equilibrium associated with barter steady state \( x(\lambda) \), any sufficiently small initial value \( z^0 > 0 \) generates a unique solution to the difference equation \( z' = f(z^{t-1}) \), and a unique equilibrium \( x = (x_t)_{t \in T} \in X \). The stationary equilibrium associated with \( x(\lambda) \) is thus not locally unique, since there is an equilibrium \( x(\lambda) \) in neighborhood \( X \) for each sufficiently small \( z^0 > 0 \). The stationary equilibrium is also not virtually locally unique because the rates of return \( R' \) vary with \( z^0 \); in particular, \( R^1 := (2 + f(z^0))/6 - z^0 \). \( \text{Q.E.D.} \)