QUASI EQUILIBRIA FOR GROWTH ECONOMIES

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We find preferences that discount utility from future consumption yet violate the standard growth condition in representative-agent endowment economies. Such economies have no competitive equilibria, but have quasi equilibria. And the supporting price systems include a type of speculative bubble. J.E.L. CLASSIFICATION NUMBERS: C6, E3.

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1. INTRODUCTION

THIS PAPER PROVES three results that find and analyze equilibrium solutions for some representative-agent endowment economies excluded by standard assumptions.

First, by extending analysis from the literature [5], we find realistic preference orders defined by sums of time-separable utility functions that discount utility from future consumption yet violate the standard growth condition (Observation 2.1). Standard proofs of the existence of equilibrium, of the equivalence of competitive and quasi equilibrium, and of the non-existence of Gilles-LeRoy speculative bubbles [2, Theorem 2] (see below) may fail for economies with such preferences because, in *every* linear topology on the commodity space, the order is discontinuous (Theorem 2.1).

Second, we prove each representative-agent endowment economy violating the growth condition has no competitive equilibrium, but does have a quasi equilibrium (Theorem 3.1).

Theorem 3.1 uses a general definition of equilibrium price systems that includes the possibility of Gilles-LeRoy speculative bubbles on durable assets (Section 3). The non-existence of competitive equilibria in Theorem 3.1 thus strengthens non-existence proofs in the literature [5] that *a priori* exclude speculative bubbles. Theorem 3.1 also contributes to the non-existence examples that include speculative bubbles, since such examples include more than one consumer, and include exotic preferences that do not discount utility from future consumption [1, 6].

Theorem 3.1 offers quasi equilibria as an alternative solution when competitive equilibria do not exist. The representative agent has positive endowment income in Theorem 3.1 , and so the quasi equilibrium is a type of ε-satisficing equilibrium for *all* positive ε.

Finally, for each representative-agent endowment economy violating the growth condition, the price system in the quasi equilibrium includes a speculative bubble (Theorem 3.1), and so contributes to the asset-pricing literature. Unlike Theorem 3.1 , bubble examples in the literature include exotic preferences that do not discount utility from future consumption [4, Section 4].

2. THE STANDARD GROWTH CONDITION

This section extends analysis from the literature [5] to find realistic preference orders that discount utility from future consumption yet violate the standard growth condition in representative-agent endowment economies. Generalizations are discussed in Section 4.

To begin, define preferences of the representative agent by an overtaking criterion (as in Equation 3 below) on the partial sums of time-separable utility functions

$$\sum_{t=1}^{T} \beta^t c_t^{1-\gamma} \tag{1}$$

over deterministic streams of a single consumption good c_t in discrete time. There, $\beta > 0$ is a felicity discount factor, and $c_t^{1-\gamma}$ is a C.E.S. felicity function with intertemporal substitution parameter $\gamma \in [0, 1)$.

Assume the agent has a positive endowment that grows at a constant exponential rate, $\lambda > -1$. The endowment is thus $y_0(1+\lambda)^t > 0$ in period t.

To fit preferences and endowments into the form most common in the general-equilibrium literature, henceforth re-normalize consumption so the endowment growth rate is zero. Specifically, let $c_t = x_t y_0 (1+\lambda)^t$ in each period, so consumption x_t measures the fraction of the endowment consumed.

Using the x_t's (rather than the c_t's) as consumption variables, the commodity endowment is now the unit vector, $\mathbf{e} = (1, 1, ...)$. Hence, restrict the commodity space to ℓ_∞, the space of bounded sequences.[2] For any consumption vector stream $\mathbf{x} = (x_t) \in \ell_\infty^+$, partial sums of utility (1) now read

$$\sum_{t=1}^{T} \beta^t (x_t y_0 (1+\lambda)^t)^{1-\gamma}$$

or, after dividing by the constant $y_0^{1-\gamma}$,

$$\sum_{t=1}^{T} \delta^t x_t^{1-\gamma}, \quad \text{for } \delta := \beta(1+\lambda)^{1-\gamma} > 0 \tag{2}$$

To cover the case where partial sums of utility (2) diverge as $T \to \infty$, define the preference of one consumption vector $\mathbf{x} = (x_t) \in \ell_\infty^+$ over another vector $\hat{\mathbf{x}} = (\hat{x}_t) \in \ell_\infty^+$ by an overtaking criterion:[3]

$$\mathbf{x} \succ \hat{\mathbf{x}} \text{ if } \liminf_T \sum\nolimits_{t=1}^{T} \delta^t (x_t^{1-\gamma} - \hat{x}_t^{1-\gamma}) > 0 \tag{3}$$

NOTE: Despite re-normalizing consumption, the utility discount factor remains β. In particular, the parameter $\delta = \beta(1+\lambda)^{1-\gamma}$ in the preference order (3) only equals the utility discount factor β in the special case when the pre-normalized endowment does not grow, $\lambda = 0$.

Standard representation and continuity properties of preferences depend on δ:

[2] Bewley [2, Section 5] shows how having endowments in ℓ_∞ means the commodity space can be reduced from the space $\Re_\infty$ of all sequences to the space ℓ_∞ of bounded sequences, without loss of generality.

[3] That criterion is different from the Weizäcker overtaking criterion. Our results hold for the latter, but only after a complex reformulation of utility representation and continuity in the first part of Theorem 2.1 .

THEOREM 2.1 *(a) If* $\delta = \beta(1+\lambda)^{1-\gamma} < 1$, *then the preference order* (3) *is represented by the Mackey continuous utility function* $u(\mathbf{x}) = \sum_{t=1}^{\infty} \delta^t x_t^{1-\gamma}$.

(b) If $\delta = \beta(1+\lambda)^{1-\gamma} \geq 1$, *then the preference order* (3) *is not representable by any utility function, and in any linear topology on the commodity space the order is neither upper semi-continuous nor lower semi-continuous.*

Because continuous utility functions are often used in standard general-equilibrium proofs, the hypothesis of Theorem 2.1 (a) is often assumed in the literature:

D.1. The standard **growth** condition is the restriction $\beta(1+\lambda)^{1-\gamma} < 1$ on utility discount rates, endowment growth rates, and intertemporal substitution rates.

OBSERVATION 2.1 *The growth condition can be violated,* $\delta = \beta(1+\lambda)^{1-\gamma} \geq 1$, *even when utility from future consumption is discounted,* $\beta < 1$.

For example, realistic parameter values like $\beta = 1/1.01$ (1 percent utility discounting), $\lambda = 0.03$ (3 percent endowment growth), and $\gamma = 0.5$ generate parameters $\delta = \beta(1+\lambda)^{1-\gamma} = \sqrt{1.03}/1.01$ greater than 1.

Proof of Theorem 2.1 . For Theorem 2.1 (a), assume $\delta = \beta(1+\lambda)^{1-\gamma} < 1$. Representation of the preference order by function $u(\mathbf{x}) = \sum_{t=1}^{\infty} \delta^t x_t^{1-\gamma}$ is obvious. And the literature proves such utility functions are Mackey $\tau(\ell_\infty, \ell_1)$ continuous [2, Theorem, p. 535].

For Theorem 2.1 (b), assume $\delta = \beta(1+\lambda)^{1-\gamma} \geq 1$, and consider three commodity vectors

$$\mathbf{e} = (1, 1, ...), \quad \mathbf{e}_1 = (1, 0, ...), \quad \mathbf{x} = (2, 0, 2, 0, ...) \tag{4}$$

The first vector is the endowment perpetuity, the second is the endowment in period 1 only, and the third is a vector alternating 1 above or 1 below the endowment.

To prove non-representation, evidently $\mathbf{e}+\mathbf{e}_1 \succ \mathbf{e}$. But $\delta \geq 1$ implies $\mathbf{e}+\mathbf{e}_1 \not\succ \mathbf{x}$ and $\mathbf{x} \not\succ \mathbf{e}$. (Those negated relations are easiest to prove in the special case $\delta = 1$ and $\gamma = 0$. In that case, $\mathbf{e}+\mathbf{e}_1 \not\succ \mathbf{x}$ because $\sum_{t=1}^{T} \delta^t ((\mathbf{e}+\mathbf{e}_1)_t^{1-\gamma} - (\mathbf{x})_t^{1-\gamma}) = 0$ for partial sums with $T = 1, 3, 5, ...$; and $\mathbf{x} \not\succ \mathbf{e}$ because $\sum_{t=1}^{T} \delta^t ((\mathbf{x})_t^{1-\gamma} - 1^{1-\gamma}) = 0$ for partial sums with $T = 2, 4, 6, ...$) So, preferences are not negatively transitive and not representable by any utility function.

To prove discontinuity, $\delta \geq 1$ implies there is an infinitesimally small marginal rate of substitution of the endowment $\mathbf{e}$ perpetuity for the endowment $\mathbf{e}_1$ in period 1 only. Precisely,

$$\mathbf{e} + (\varepsilon\mathbf{e} - \mathbf{e}_1) \succ \mathbf{e} \text{ and } \mathbf{e} \succ \mathbf{e} - (\varepsilon\mathbf{e} - \mathbf{e}_1), \quad \text{for every } \varepsilon \in (0, 1] \tag{5}$$

In particular, in any linear topology on the commodity space, the preference order is not upper semi-continuous because, if it were, order $\mathbf{e} \succ \mathbf{e} - \mathbf{e}_1$ would imply $\mathbf{e} \succ \mathbf{e} + (\varepsilon\mathbf{e} - \mathbf{e}_1)$, for some $\varepsilon \in (0, 1]$, which contradicts $\mathbf{e} + (\varepsilon\mathbf{e} - \mathbf{e}_1) \succ \mathbf{e}$ (5). Likewise, $\mathbf{e} \succ \mathbf{e} - (\varepsilon\mathbf{e} - \mathbf{e}_1)$ (5) implies the preference order is not lower semi-continuous. *Q.E.D.*

3. COMPETITIVE AND QUASI EQUILIBRIUM

This section proves each economy violating the growth condition has no competitive equilibrium, but does have a quasi equilibrium. And the price system in that quasi equilibrium includes a type of speculative bubble.

To be precise, consider any representative-agent endowment economy $\mathcal{E} = (\succ, \mathbf{e})$, with preferences (3) defined by parameters $\delta = \beta(1 + \lambda)^{1-\gamma} > 0$ and $\gamma \in [0, 1)$ and with the unit endowment $\mathbf{e} = (1, 1, ...)$. The following are standard definitions:

D.2. A **price system** is a positive linear functional π over ℓ_∞ such that $\pi\mathbf{e} > 0$.
D.3. A **competitive** equilibrium is a price system π such that each preferred consumption $\mathbf{x} \succ \mathbf{e}$ has cost $\pi\mathbf{x} > \pi\mathbf{e}$.
D.4. A **quasi** equilibrium is a price system π such that each preferred consumption $\mathbf{x} \succ \mathbf{e}$ has cost $\pi\mathbf{x} \geq \pi\mathbf{e}$.

Quasi equilibria are most often used in the literature as an intermediate step toward finding a competitive equilibrium. But since quasi equilibria exist more generally, we seek to interpret a quasi equilibrium as a solution in its own right. To that end, since endowment income $\pi\mathbf{e} > 0$ (D.2), the inequality "$\pi\mathbf{x} \geq \pi\mathbf{e}$" in the definition of quasi equilibrium (D.4) is equivalent to "$\pi\mathbf{x} > (1 - \varepsilon)\pi\mathbf{e}$ for every $\varepsilon > 0$". Hence, a quasi equilibrium has the following ε-satisficing interpretation: each preferred consumption $\mathbf{x} \succ \mathbf{e}$ has cost $\pi\mathbf{x} > (1 - \varepsilon)\pi\mathbf{e}$, for each tolerance $\varepsilon > 0$. That is, the consumer cannot find a preferred consumption if he spends only the fraction $(1 - \varepsilon)$ of his endowment income $\pi\mathbf{e}$. Such satisficing behavior is justified by the usual arguments for bounded rationality [7], such as the consumer incorrectly computing his endowment income as $(1 - \varepsilon)\pi\mathbf{e}$ rather than $\pi\mathbf{e}$. And the usual objection to bounded rationality (that solutions depend on the elusive parameter ε) does not apply because a quasi equilibrium is an ε-satisficing equilibrium for every $\varepsilon > 0$.

THEOREM 3.1 (*a*) *If* $\delta < 1$, *then the economy* $\mathcal{E} = (\succ, \mathbf{e})$ *has the linear functional* $\pi\mathbf{x} = \sum_{t=1}^{\infty} \delta^t x_t$ *as a competitive equilibrium and quasi equilibrium. And every other competitive and quasi equilibrium is a positive multiple of* π.

(*b*) *If* $\delta \geq 1$, *then the economy has no competitive equilibrium, but does have a quasi equilibrium, for the price system functional defined by a Banach limit* $\mathrm{Lim}_{T\to\infty}$:

$$\pi\mathbf{x} := \mathrm{Lim}_{T\to\infty} \frac{\sum_{t=1}^{T} \delta^t x_t}{\sum_{t=1}^{T} \delta^t} \tag{6}$$

(Technical note: In the quasi-equilibrium price system (6), a Banach limit $\mathrm{Lim}_{T\to\infty} y_T$ is a particular type of positive linear functional over the space ℓ_∞ of bounded sequences. A Banach limit extends the standard limit functional $\lim_{T\to\infty} y_T$ from the subspace of convergent sequences. In particular, the Banach limit in the price system (6) is well defined because, for each bounded sequence of consumption x_t, the right-hand-side terms $y_T := \sum_{t=1}^{T} \delta^t x_t / \sum_{t=1}^{T} \delta^t$ (6) are also a bounded sequence. Evidently $\pi\mathbf{e} = 1$, and the functional $\pi\mathbf{x}$ (6) is positive and linear (D.2) because the Banach limit is a positive linear functional.)

The existence and uniqueness of equilibrium prices (up to normalization) in Theorem 3.1 (a) is standard in representative-agent endowment economies when the preference order is represented by utility $u(\mathbf{x}) = \sum_{t=1}^{\infty} \delta^t x_t^{1-\gamma}$.

Theorem 3.1 (b) offers quasi equilibria (and ε-satisficing equilibria) as an alternative solution when competitive equilibria do not exist. Note, $\delta \geq 1$ implies the price system in that quasi equilibrium (6) values the endowment perpetuity $\mathbf{e}$ at $\pi\mathbf{e} = 1$ while the value of the endowment in each individual period is zero. Gilles and LeRoy interpret such price systems as a pure speculative bubble, and they offer examples [4, Section 4]. Like our Theorem 3.1 (b), Gilles-LeRoy bubbles are possible in their example economies because there is an infinitesimally small marginal rate of substitution of the endowment $\mathbf{e}$ perpetuity for the endowment in an individual period only (5). But unlike our Theorem 3.1 (b), Gilles-LeRoy generate infinitesimally small marginal rates by considering the exotic utility function $u(\mathbf{x}) = \liminf_t x_t$, which does not discount utility from future consumption [4, Section 4].

The non-existence of competitive equilibria in Theorem 3.1 (b) strengthens non-existence proofs in the literature [5], [6, Example 8.2] that *a priori* exclude speculative bubbles in the Gilles-LeRoy sense. Theorem 3.1 (b) also contributes to the non-existence examples that include speculative bubbles, since such examples include more than one consumer, and include exotic utility functions like $u(\mathbf{x}) = \liminf_t x_t$ [6, Example 6.2] or a Banach limit $u(\mathbf{x}) = \mathrm{Lim}_t x_t$ [1], which do not discount utility from future consumption.

Proof of Theorem 3.1 (b). Suppose $\delta \geq 1$ in the preference order (3).

For the proof of non-existence, suppose there were a competitive equilibrium π, and consider again the endowment perpetuity $\mathbf{e} = (1, 1, ...)$ and the endowment $\mathbf{e}_1$ in period 1 only (4). As before, $\delta \geq 1$ implies, for every $\varepsilon \in (0, 1]$, $\mathbf{e} + \mathbf{e}_1 \succ \mathbf{e}$ and $\mathbf{e} + (\varepsilon\mathbf{e} - \mathbf{e}_1) \succ \mathbf{e}$ (5). Hence, budget-constrained maximization (D.3) implies $\pi\mathbf{e}_1 > 0$ and $\pi(\varepsilon\mathbf{e} - \mathbf{e}_1) > 0$. But the latter inequalities holding for small ε imply $\pi\mathbf{e}_1 \leq 0$, which contradicts the former inequality.

For the proof of existence, the concavity of felicity $v(x_t) := x_t^{1-\gamma}$, its value $v(1) = 1$, and its derivative $v'(1) = 1$ imply $x_t^{1-\gamma} - 1 \leq x_t - 1$. Hence, the partial sums

$$\sum_{t=1}^{T} \delta^t (x_t^{1-\gamma} - 1) \leq \sum_{t=1}^{T} \delta^t (x_t - 1)$$

But for each preferred consumption $\mathbf{x} \succ \mathbf{e}$, the left-hand side $\sum_{t=1}^{T} \delta^t (x_t^{1-\gamma} - 1)$ is positive (3) for large T, which implies the right-hand side $\sum_{t=1}^{T} \delta^t (x_t - 1)$ is also positive for large T, which with the definition of the price system (6) implies $\pi \cdot (\mathbf{x} - \mathbf{e}) \geq 0$, and so $\pi\mathbf{x} \geq \pi\mathbf{e}$. Hence, π is a quasi equilibrium (D.4). *Q.E.D.*

4. CONCLUSION

We offer Theorem 3.1 (b) on the existence of quasi and satisficing equilibria to provoke further results finding equilibrium solutions for more general economies currently excluded from the general-equilibrium theory of competitive equilibrium. One known result is the generalization of a constant utility discount factor β and a constant endowment growth rate γ. It turns out non-constant discounting and growth generates a sequence of parameters $\delta_1, \delta_2, \ldots$ that replaces the exponential sequence $\delta^1, \delta^2, \ldots$ in the definition of preferences (3) and in Theorem 2.1 and Theorem 3.1 , with the condition $\sum_{t=1}^{\infty} \delta_t < \infty$ replacing the growth condition $\delta < 1$. We choose, however, to leave such generalizations to later work that also allows heterogeneous agents. As a first step, Burke [3] allows heterogeneous agents and finds a type of ε-satisficing equilibria, but with a fixed tolerance $\varepsilon > 0$.

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