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Robust asset prices with bubbles

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Abstract

We generally establish equilibrium asset prices than can include price bubbles yet (a) be robust to truncations of the economy and (b) exclude trade in non-consumables, like money, stock certificates, or land deeds.

Keywords: Asset; Bubbles; Robust

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1. Introduction

Well-known examples suggest that equilibrium among overlapping generations can allow the market price of an asset, like money, stock, or land, to exceed the present value of its dividends (Samuelson, 1958; Tirole, 1985). Allowing such asset-price bubbles makes overlapping generations useful, but causes problems for extending Arrow–Debreu general-equilibrium theory and methods from truncated (finite-population, finite-horizon) economies. (a) The price of money jumps to zero if the time horizon is truncated from infinity, no matter how distant the final period. Thus, positive money prices are not robust. (b) Price bubbles can entail trade in non-consumables, like money, stock certificates, or land deeds, which complicate the model with extra variables, and makes counterfeiting possible.

We face these problems for overlapping-generations economies with two theorems. Theorem 1: there exists a limit equilibrium of a sequence of truncated subeconomies. Thus, such a limit equilibrium generally establishes a robustness benchmark for other equilibria. For instance, in some well-known examples, limit-equilibrium asset prices include bubbles, but when the time horizon is truncated, any discontinuous drop in the bubble part of the asset prices is balanced by a discontinuous rise in the present value of dividends, making the change in asset prices continuous. Theorem 2: a limit equilibrium allows those asset-price bubbles to be incorporated into a price system defined (exclusively) over the space of consumable

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commodities. Theorems 1 and 2 combined strengthen the conclusions of equilibrium–existence theorems for infinite-lasting assets in the literature (Burke, 1988; Wilson, 1981) because our asset prices are robust and our bubbles exclude non-consumables. Finally, an original non-existence example shows that Theorem 2 fails for family dynasties.

2. Assumptions and definitions

This section states our theorems for discrete-time, overlapping-generations economies excluding production and immortal family dynasties but allows infinite-lasting assets with constant dividends, like land or money (which have zero dividends).

There are an infinite number of periods, $t = 1, 2, \ldots, n_t$ goods per period; two (consecutive) periods per lifetime; and, altogether, an infinite number of consumers, $i = 1, 2, \ldots$. Each consumer living in periods t and $t+1$ has a preference order $>_i$ represented by a continuous, non-decreasing, quasi-concave utility function u_i over lifetime consumption $x_i = (x_i^t, x_i^{t+1})$ in $X_i := (\mathbb{R}_+^{n_t} \times \mathbb{R}_+^{n_{t+1}})$. Each consumer is endowed with an asset vector $e_i = (0, \ldots, 0, e_i^t, e_i^{t+1}, \ldots)$ of dividends in the commodity space $\Pi_1^\infty \mathbb{R}_+^{n_t}$. Dividends may succeed the consumer, like land. Call an endowment e_i *finite lasting* if dividends $e_i^\tau \neq 0$ for only a finite number of periods τ. An *allocation* of consumption $x = (x_i) \in X := \Pi_i X_i$ balances commodity materials, $\Sigma_i x_i^t = \Sigma_i e_i^t$, at each period t. (Here we sum consumption over consumers living in period t; endowments, over consumers with endowments in period t.) Assume the economy is irreducible at each allocation (Burke, 1988, p. 285).

At each period T, we truncate each overlapping-generations economy $\mathscr{E} = \{>_i, e_i\}$ by setting all consumption and endowments to zero after period T. The truncated economy, $\mathscr{E}^T$ inherits all assumptions about overlapping-generations economies, except irreducibility. We simply assume some sequence $\mathscr{E}^{T^\alpha}$ of truncated subeconomies is irreducible at each truncated allocation x in $X^T \subset X$.

Following Arrow–Debreu, an *equilibrium* for the truncated economy $\mathscr{E}^T$ is a truncated allocation x in X^T and a sequence $p = (p^1, \ldots)$ in $P := (\Pi_1^\infty \mathbb{R}_+^{n_t}) \backslash \{0\}$ of present-value prices so that each consumption maximizes utility subject to a lifetime budget constraint

$$p^t \cdot x_i^t + p^{t+1} \cdot x_i^{t+1} \leqq p^t \cdot e_i^t + \cdots + p^T \cdot e_i^T ,$$

for consumers living before period T, and $p^T \cdot x_i^T \leqq p^T \cdot e_i^T$, for consumers living only in period T.

3. A limit equilibrium

From existence proofs for a competitive equilibrium in the literature (Wilson, 1981, section 4), we extract the general existence of a limit equilibrium as follows:

Theorem 1 (Wilson). *Each overlapping-generations economy $\mathscr{E}$ has a limit equilibrium (x, p) in $X \times P$ in the sense that some sequence $\mathscr{E}^{T^\alpha}$ of truncated subeconomies has Arrow–Debreu* *equilibria* (x^α, p^α) *for which each consumers's consumption converges,* $x_i^\alpha \to x_i$*; and each period's price converges,* $p^{t\alpha} \to p^t$.

By definition, consumption quantities and commodity prices in a limit equilibrium are robust to finite truncations of the time horizon. We conclude that asset prices are also robust to truncation, even for infinite-lasting assets, like land, and even in examples where asset prices necessarily include bubbles (Tirole, 1985, proposition 1c, and Wilson, 1981, section 7). Hence, when the time horizon is truncated, any discontinuous drop in the bubble part of the asset prices is balanced by a discontinuous rise in the present value of dividends, making the change in asset prices continuous. This is stated formally as follows.

Corollary. In each limit equilibrium (x, p) *the price of each consumer's asset converges across the truncated equilibria,* $(p^{t,\alpha} \cdot e_i^t + \cdots + p^{T^\alpha,\alpha} \cdot e_i^{T^\alpha}) \to w_i$ *for some* w_i.

Thus, a limit equilibrium generally establishes a robustness benchmark. In particular, a limit equilibrium excludes positive money prices as not being robust because money prices are zero in each truncated equilibrium. Another benefit of a limit equilibrium is its approximation as an equilibrium for a finite population and a finite horizon, where equilibria generically obey efficiency, determinacy, and stability properties not generically found among overlapping generations.

Proof of Corollary. For each consumer, looking across the truncated equilibria, the convergence of price and consumption in Theorem 1 implies the convergence of expenditure, $(p^{t,\alpha} \cdot x_i^{t,\alpha} + p^{t+1,\alpha} \cdot x_i^{t+1,\alpha}) \to w_i$ for $w_i := p^t \cdot x_i^t + p^{t+1} \cdot x_i^{t+1}$. But budget constraints hold with equality in a subequilibrium, meaning that asset prices equal expenditure. Hence, asset prices also converge. □

4. Bubbles in the price system

Theorem 2. At each limit equilibrium (x, p), *there exists an additive, linear-homogeneous function* $z \mapsto b(z)$ *of the commodity space into the extended line* $[0, +\infty]$, *under which each consumption maximizes utility subject to the budget constraint*

$$p^t \cdot x_i^t + p^{t+1} \cdot x_i^{t+1} \leqq (p^t \cdot e_i^t + \cdots) + b(e_i),$$

where $b = 0$ *over finite-lasting endowments and all other finite-lasting vectors in the commodity space.*

When combined with the existence of a limit equilibrium (Theorem 1), Theorem 2 strengthens the conclusions of equilibrium–existence theorems for infinite-lasting assets in the literature (Burke, 1988; Wilson, 1981) because our asset prices are robust and our bubbles exclude non-consumables. For comparison, our bubbles in the price system are like the bubbles in Gilles (1989), except that our asset prices with bubbles prove robust to truncation.

Proof. Wilson (1981, section 4) proves that limit consumption for each consumer maximizes utility subject to the budget constraint

$$p^t \cdot x_i^t + p^{t+1} \cdot x_i^{t+1} \leqq w_i,$$

as defined by the limit of wealth in the corollary. It remains to find a bubble function that satisfies the stated properties in Theorem 2, plus the wealth constraint

$$w_i = (p^t \cdot e_i^t + \cdots) + b(e_i).$$

To define the bubble at all vectors for each z in the commodity space, we compute the value $\Sigma_1^T p^{t\alpha} z^t$ under subequilibrium prices p^α. Since the value is non-negative, it is contained in the (compact) interval $[0, +\infty]$ of extended real numbers. Hence, the Tychonoff theorem extracts a subnet (x^α, p^α) of the original sequence of subequilibria under which, for each z, the truncated values have a limit, $\lim_\alpha \Sigma_1^{T^\alpha} p^{t\alpha} z^t$. Hence, we define the bubble function as

$$b(z) := \begin{cases} \lim\limits_{\alpha} \sum\limits_{1}^{T^\alpha} p^{t\alpha} z^t - \sum\limits_{1}^{\infty} p^t z^t, & \text{if} \sum\limits_{1}^{\infty} p^t z^t < +\infty, \\ +\infty, & \text{if} \sum\limits_{1}^{\infty} p^t z^t = +\infty. \end{cases}$$

The constraint $w_i = (p^t \cdot e_i^t + \cdots) + b(e_i)$ follows from the wealth convergence $w_i = \lim_\alpha \Sigma_1^{T^\alpha} p^{t\alpha} e_i^t$ in the corollary.

The non-negativity of the bubble function follows because the non-negativity and convergence of prices imply that the limit operation is upper semi-continuous, $\lim_\alpha \Sigma_1^{T^\alpha} p^{t\alpha} z^t \geqq \Sigma_1^\infty p^t z^t$. The function $z \mapsto b(z)$ is clearly additive, liner-homogeneous, and zero over finite-lasting vector. □

5. Non-existence example

This section contains an original non-existence example showing that Theorem 2 fails for family dynasties. There is one good per period and one mortal consumer per generation: consumer t has endowment $e_t = (0, \ldots, e_t^t = 1, e_t^{t+1} = 1, 0, \ldots)$ and utility $u_t(x) = x^t + 3x^{t+1}$. There is one immortal consumer (dynasty), with endowment $e_0 = (2^{-1}, \ldots, 2^{-t}, \ldots)$ and utility $u_0(x) = x^1 + \Sigma_2^\infty \min\{x^t, 2^{-t-1}\}$. (Note period $t > 1$ satiation at $x_0^t = 2^{-t-1}$.)

Theorem 3. The example economy satisfies routine extensions of all assumptions in this paper to immortal consumers. And the economy has a limit equilibrium (in an extended sense).

Yet there does not exist an allocation x, price system p, and bubble function b (as in Theorem

2) so that each mortal consumer maximizes utility subject to his/her usual budget constraint, and the immortal consumer maximizes subject to

$$(p^1 \cdot x_0^1 + p^2 \cdot x_0^2 + \cdots) + b(x_0) \leqq (p^1 \cdot e_0^1 + p^2 \cdot e_0^2 + \cdots) + b(e_0). \tag{1}$$

Proof. For proof by contradiction, suppose that there were such an equilibrium (x, p, b). Throughout, monotonicity implies that all prices are positive, $p^t > 0$.

Since consumer 0 is insatiable (utility increases in good 1), utility maximization implies finite wealth, $\Sigma_1^\infty p^t e_0^t = \Sigma_1^\infty 2^{-t} p^t < \infty$. In particular, positive prices imply $3p^t > p^{t+1}$ for an infinite number of t. Fixing any of those $t > 1$, $3p^t > p^{t+1}$ implies that consumer t chooses zero young-age consumption, $x_t^t = 0$. But $p^t > 0$ implies that consumer 0 only consumes goods in period t up to the satiation of utility, $x_0^t \leqq 2^{-t-1}$. Therefore, a material balance implies that consumer $t-1$ demands the remaining supply of good t,

$$x_{t-1}^t \geqq e_0^t + e_{t-1}^t + e_t^t - x_0^t = 2 + 2^{-t} - x_0^t > 2\,.$$

However, consumer $t-1$ can only afford such old-age consumption if prices fall, $p^{t-1} > p^t$. In particular, $3p^t > p^{t+1}$ implies that $3p^{t-1} > p^t$, which, after iteration, implies that, for all periods preceding t, $3p^\tau > p^{\tau+1}$ and $p^{\tau-1} > p^\tau$ (because $3p^\tau > p^{\tau+1}$ implies $p^{\tau-1} > p^\tau$). Hence, prices fall for all periods preceding t and, since there is an infinite number of such t, prices fall in every period, $p^{t-1} > p^t$.

In particular, $p^1 > p^2$ implies that consumer 1 chooses zero young-age consumption, which with a material balance implies that consumer 0 consumes the total endowment, $x_0^1 = e_0^1 + e_1^1 = 1 + 2^{-1} > 1$. But falling prices imply the present value of consumer 0's dividends $\Sigma_1^\infty p^t e_0^t = \Sigma_1^\infty 2^{-t} p^t < p^1$. Therefore, consumer 0's budget constraint (1) can hold only if there is a positive bubble, $b(e_0) > 0$. But falling prices with a positive bubble imply that there is no consumption that maximizes consumer 0's utility subject to his budget constraint (1).

On the one hand, falling prices put a lower bound, $x_0 \geqq 1/2e_0$, on any alleged solution. The previous paragraph proves the bound for the first period. For later periods, $x_0^t \geqq 1/2e_0^t$ means that consumption satiates utility. That follows because, at any consumption before satiation, the marginal rate of substitution $\partial_t u_0 / \partial_1 u_0 = 1$, while falling prices imply a relative price, $p^t/p^1 < 1$.

On the other hand, the endowment bubble $b(e_0) > 0$ implies that the consumption bubble $b(x_0) > 0$, for an alleged solution $x_0 \geqq 1/2e_0$, which will contradict budget-constrained utility maximization. For a proof, normalize $p^1 = 1$, fix any period T, and consider alternative consumption with an increase in period 1 and truncation after period T, $\hat{x} = (x_0^1 + b(x_0), x_0^2, \ldots, x_0^T, 0, \ldots)$. That alternative consumption costs

$$(p^1 \cdot \hat{x}_0^1 + p^2 \cdot \hat{x}_0^2 + \cdots) + b(\hat{x}_0) = (p^1 \cdot x_0^1 + \cdots + p^T \cdot x_0^T) + b(x_0),$$

which is less than the cost $(p^1 \cdot x_0^1 + p^2 \cdot x_0^2 + \cdots) + b(x_0)$ of the alleged solution. But computing the difference

$$u_0(\hat{x}) - u_0(x_0) = b(x_0) - \Sigma_{T+1}^\infty \min\{x_0^t, 2^{-t-1}\} \geqq b(x_0) - 2^{-T-1}$$

in utility shows that, for sufficiently large T, the alternative consumption yields higher utility than the alleged solution, which contradicts budget constrained utility maximization. □

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