Existence of a Pareto-optimal equilibrium in nearly-stationary overlapping-generations economies*

Jonathan L. Burke
Economics Department, University of Texas, Austin, TX 78712-1173, USA

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Summary. In overlapping-generations models of fiat money, the existence of a Pareto-optimal equilibrium — which defines an optimal quantity of money — is more general than well-known counter-examples suggest. Those examples, having no optimal equilibrium just because there are small variations in households’ tastes and endowments across generations, are not typical. On the contrary: For an open-dense, full-measure subset of smooth stationary economies and an open-dense subset of continuous stationary economies, introducing small variations in tastes and endowments across generations preserves the existence of an optimal equilibrium. Put simply, optimal equilibria generically exist for nearly-stationary economies.

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1. Introduction

Most applications of the overlapping-generations model of fiat money\(^1\) ensue from a Pareto-optimal equilibrium; such an equilibrium defines an optimal quantity of money. While the existence of a Pareto-optimal equilibrium is proved in Samuelson’s original analysis [23], his proof uses many simplifying assumptions: there is only pure-exchange; only one type of consumption good in each period; only 2 or 3 periods in each lifetime; only one type of household in each generation; and only one type of generation — that is, all taste and endowment profiles are stationary across generations, merely shifted out in time.

The importance of finding an optimal equilibrium naturally created interest in finding weaker, and more realistic, sufficient assumptions. To date, almost every assumption has been significantly weakened: there can be production [11]; many types of consumption goods in each period [6, 11, 17, 22]; many periods in each lifetime [11]; and many types of households in each generation [11, 17, 22]. But one assumption remains: all taste and endowment profiles are stationary across generations. Hence, while current models may approximate the position of a

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\(^1\) Throughout, “money” means some intrinsically-valueless durable good in fixed supply.
monetary economy at a moment in time, stationarity rules out any application to
the basic macroeconomic policy analysis of stabilizing short-run-equilibrium fluctua-
tions in an evolving and changing economic environment.2

The single published analysis of non-stationary overlapping-generations models
of fiat money contains only counter-examples to the existence of a Pareto-optimal
equilibrium. In their most striking example, Cass–Okuno–Zilcha [12] model an
economy that has no optimal equilibrium just because there are small variations in
households’ tastes and endowments across generations. Building up that work,
Millán [19, 20] shows that indifference curves, which in the original example come
arbitrarily close to exhibiting perfect complements (kinks), can be made smooth.
And Burke [7] shows that the variations across generations, which in the original
example include both endowments and preferences, need only involve endowments,
and that variations can vanish exponentially.

None of those counter-examples settle the existence issue, however, because it
is never shown that any of them are robust. That is, it is never shown that the set of all
counter-examples, in some ambient space of alternative economic models, has
positive measure or has a non-empty interior. Hence, questions remain: Do
robust counter-examples exist? Or are there no robust counter-examples? and an
optimal equilibrium exists for a generic subset of economies.

So far, the only answers are in a decade-old unpublished working paper and
a Ph.D. thesis from Millán [19, 20]. First, Millán forms an ambient space of
alternative economic models that includes all the smooth refinements of the original
Cass–Okuno–Zilcha non-existence example: there is only pure-exchange, only one
type of consumption good in each period, only 2 periods in each life-time, and only
one type of generation; but many types of households in each generation. He then
formulates and proves this:

Existence Proposition. For a generic subset of stationary economies, introducing
small variations in tastes and endowments across generations preserves the
existence of an optimal equilibrium. (Put simply, optimal equilibria generically exist
for nearly-stationary economies.)

The point of the current paper is to make accessible a concise statement of that
Existence Proposition, generalize the proposition past Millán’s original version, and
simplify Millán’s original proof.

We first generalize the ambient space of stationary economies. Millán worked
only with continuously-twice-differentiable preference orders and utility functions.
On that space our results agree: “generic” means, for fixed preferences, the subset
of stationary economies is open-dense and of full measure. But I also work with
continuous preference orders and utility functions. On that space, “generic” means
the subset of stationary economies is open-dense.

Our second generalization allows significantly more variation in tastes and
endowments. After proving the existence proposition easily in the special case where

2 See Balasko-Shell [3] for some thoughtful arguments for the analysis of money in non-stationary
economies.
there is only one type of household in each generation, Millán proves existence for the general many-household economy by passing back and forth between it and an ad-hoc single-household "economy". It turns out that trick requires that "indifference curves" in the ad-hoc "economy", constructed from offer curves in the original economy, must have their curvature bounded from above. Passing back to the original economy, that constrains the curvature of offer curves -- which, in turn, constrains the first, second, and third derivatives of the original utility functions. While Millán asserts that such derivative constraints seem innocuous, even for very non-stationary economies, his only precise assertion is that the derivative constraints are satisfied when variations in tastes (measured by differences in utility function values and first, second, and third derivatives) and variations in endowments converge to zero across generations. Of course, convergence rules out a perpetually evolving and changing economic environment.

In our formulation of the Existence Proposition, non-stationarity still needs to be small, less than a certain positive tolerance at each time, but it need not vanish across generations. Thus, the modeled world can be perpetually evolving and changing. And that non-stationarity is measured with minimal structure: variations in utility functions are measured only by their values -- any derivatives, and there need not be any, are unrestricted.

For a simple proof of our reformulated existence proposition, we follow the general methods of Kehoe–Levine [17], solving market-clearing equations as a system of first-order difference equations in equilibrium prices, or discount factors. Generically, there are two cases. One: That equation system is a forward contraction (shifts in today's factors lead to smaller shifts in tomorrow's factors), and so we solve it by forward substitution. Two: That equation system is a backward contraction (anticipated shifts in tomorrow's factors lead to smaller shifts in today's factors), and so we solve it by backward substitution. In either case, the contractions dampen variations in the system, even perpetual variations, and so guarantee a solution.

Before offering the plan for the rest of this paper, we confess four special features of our current analysis that make it fall short being the final resolution of the existence issue. But as we identify each feature, we defend it and argue that the current analysis has enough to say to warrant attention.

Special Feature 1: There is just one type of consumption good in each period and just two periods in each lifetime. In defense, a proof of a many-good, many-period existence proposition can be based on the simple one-good, two-period proof presented in this paper. (The details are in another paper [8].) The key to that generalization is that a generic system of market-clearing equations in \( n \) goods splits, after a change of local coordinates, into \( n \) independent systems in 1 good.

Special Feature 2: Each economy must be nearly stationary. Defense: First, nearly-stationary economies are sufficiently general for applications of stabilizing short-run-equilibrium fluctuations in an evolving and changing economic environment. Second, since we don't know whether a general form of the existence proposition is true, the analysis of the existence problem for largely-nonstationary economies involves a search for non-existence examples. To guide that search, our generic existence proof for nearly-stationary economies shows that an open (robust) set of counter-examples can only be found among largely-nonstationary economies,
not among the nearly-stationary economies where Cass–Okuno–Zilcha, and their followers, looked.

Special Feature 3: Each consumption set is a positive orthant, and there is no production. Defense: Adding the usual type of closed, convex, bounded-below consumption sets would be routine, as would adding strictly-convex production sets that each span only a finite number of periods. But since the general existence issue is still unsettled for largely-nonstationary pure-exchange economies – Do robust counter-examples exist? Or are there no robust counter-examples? and an optimal equilibrium exists for a generic subset of economies – it seems best to hold off any full analysis of production.

Special Feature 4: The supply of fiat money is fixed. Defense: It is trivial, for the simple class of economies we consider, to generalize our existence result to money-supply trajectories that initially vary, but are eventually fixed after some point in time. Thus, our result solves part (but only part) of the general problem of classifying which money-supply trajectories are part of a Pareto-optimal equilibrium. We don’t even try to solve the rest of the classification problem since it is not yet solved for stationary economies, and this paper is about the particular complications of non-stationarity.

Here is the simple plan of the rest this paper: Section 2 recapitulates the usual overlapping-generations model of fiat money. Section 3 deals with some technical details involved in identifying those equilibria that are Pareto optimal. Section 4 sets up the assertion of the generic existence of an optimal equilibrium for nearly-stationary economies; Section 5 contains the proof.

2. The model

The following is a brief list of the definitions comprising the overlapping-generations model of fiat money. We include only enough detail to formulate and prove our existence proposition, leaving extensive descriptions and interpretations to others [1, 2, 6, 15, 16]. The economy features only one type of consumption good in each period and two periods in each lifetime, but many types of households in each generations.

1. Time and goods: Since there is only one type of good in each period, time and goods have the same index: good \( t \) is the good available at time \( t \) \( (t = 1, 2, \ldots) \).

2. Households: Since each household \( h \) \( (h = 1, \ldots, H) \) of each generation \( t \) \( (t = 1, 2, \ldots) \) is alive long enough to consume just two goods, \( t \) and \( t + 1 \), describe the household by a commodity endowment \( e^{th} = (e^t, e^{t+1}) \) in \( \mathbb{R}^2_+ \) and a strictly-increasing, strictly-quasi-concave, continuous utility function \( u^h \) defined over all

\[ \text{Indeed, aside from positive results for fixed or eventually-fixed money supplies, Okuno and Zilcha} \]

\[ \text{[21]} \]

\[ \text{give the only other result on the classification problem: Pareto-optimal equilibria are often} \]

\[ \text{impossible when the money supply grows exponentially [21, Theorem 4] (in that case, for money to be} \]

\[ \text{exchanged between generations with bounded commodity endowments, the real value of money must} \]

\[ \text{be bounded over time, which requires commodity prices to grow exponentially and, thus, violate the} \]

\[ \text{Cass price-characterization of optimality).} \]
consumptions \( c^h = (c^h_i, c^h_n) \) in \( \mathbb{R}^2_+ \). And, to avoid complicated boundary behavior, assume that each household prefers its endowment \( e^h \), or any other consumption in \( \mathbb{R}^2_+ \), to any consumption on the boundary of \( \mathbb{R}^2_+ \).

3. Generations and economies: A pair of \( H \)-lists

\[
(e^h, u^h) := ((e^1, u^1), \ldots, (e^H, u^H))
\]

of household endowments and utility functions defines a generation. And a sequence

\[
\mathcal{E} := ((e^1, u^1), \ldots, (e^h, u^h), \ldots)
\]

of \( H \)-lists defines an economy.

4. Stationarity: An economy that is a constant sequence, \((e^h, u^h) = (e, u)\), is called stationary. That is, all generations are alike, merely shifted out in time. (By a routine re-scaling of commodities, stationarity allows exponentially-growing endowments, or exponentially-growing populations of households with homothetic utility.)

5. Discount factors: Each household \( h \) of each generation \( t \) is a price-taker, and so faces a linear budget constraint

\[
c^h_t + d_t c^h_{t+1} \leq e^h_t + d_t e^h_{t+1}
\]

There, \( d_t > 0 \) is the relative price at time \( t \) of next-period consumption, the discount factor.

6. Demand: Since the utility function \( u^h \) of each household is strictly-increasing, strictly-quasi-concave, and continuous, it can be uniquely maximized subject to the household's budget constraint. Within any generation \( t \), adding up those consumptions and subtracting off endowments across households defines lifetime excess demand \((y_t, z_{t+1}) = \sum_h (c^h - e^h) \) in \( \mathbb{R}^2 \). (\( y_t \) is the excess demand for good \( t \), when the generation is young; \( z_{t+1} \) is the excess demand for good \( t + 1 \), when the generation is old.)

While that demand depends on many things, we only recognize it functionally dependent on the discount factor \( y_t(d_t), z_{t+1}(d_t) \), with shifts in preferences and endowments modeled as shifts in that functional relationship. Since each household's utility function \( u^h \) is strictly-increasing, strictly-quasi-concave, and continuous, each excess-demand function \((y_t, z_{t+1}): \mathbb{R}^2_+ \rightarrow \mathbb{R}^2 \) is continuous, homogeneous of degree 0, and satisfies Walras's law: \( y_t(d_t) + d_t z_{t+1}(d_t) = 0 \).

7. Monetary endowments and market clearing for the first good: This is an economy with fiat money; or, at any rate, some intrinsically-valueless durable good in fixed supply. All money is initially entirely owned by people already alive in the first period; call them the initially old. Their money may or may not be assigned a positive exchange value in equilibrium, but money cannot have a negative exchange value, due to the possibility of free disposal.

The initially old live out their lives in period 1, and so only consume good 1. Assuming that their endowment of good 1 is in their consumption set and that their preferences for good 1 are locally insatiable, the initially old's excess demand for good 1 equals the exchange value of their money endowment, and so is either positive (if the exchange value of money is positive) or zero (if the exchange value of money is zero), but is never negative. Thus, market clearing for good 1 reduces
to an inequality
\[ y_t(d_t) \leq 0 \] (1)
in the excess demand by the generation of consumers that are young at time 1.

8. Equilibrium: For any economy &E, an equilibrium is a sequence
\[ (d_1, \ldots, d_{t-1}, d_t, \ldots) \]
of discount factors satisfying the special market-clearing inequality (1) for the first good, and the usual market-clearing equality
\[ z_t(d_{t-1}) + y_t(d_t) = 0 \] (2)
for every other good t. There, the excess demand by the currently old (generation \( t - 1 \), which faces the discount factor \( d_{t-1} \)) plus the excess demand by the currently young (generation \( t \), which faces the discount factor \( d_t \)) equals 0.

9. Optimality: An equilibrium is Pareto optimal if there is no reallocation of consumptions that increases the utility of some households without decreasing the utility of the other households.

3. A price-characterization of optimality

To help find an equilibrium that is optimal, note the general form of any potential reallocation of consumptions that increases the utility of some households without decreasing the utility of the other households: the young give to the old at each time, with the initially old receiving but not giving. That clearly increases the utility of the initially old, and it does not decrease the utility of the other households when the decrease in utility from having less young-age consumption is made up by the increase in utility from having more old-age consumption. Thus, such improving reallocations require that the marginal utilities of old-age consumptions must be, on average, sufficiently larger than the marginal utilities of young-age consumptions. Since, in equilibrium, such relative marginal utilities are revealed by the market discount factors, we have this characterization: an equilibrium is not optimal if and only if, on average, discount factors are sufficiently high. Put in a positive form, an equilibrium is optimal if and only if, on average, discount factors are sufficiently low.

Precise price characterization results – which specify “on average” and “sufficiently low” – began with Cass [10], for simple capital-accumulation models. They were easily extended [1, 4, 5, 9, 16, 21] to include most of the overlapping-generations economies considered here. A key condition to apply the price characterization restricts, for each household, the Gaussian curvature \( C^{th}(c^{th}) \) of the indifference curve of \( u^{th} \) passing through consumption \( c^{th} \) in \( \mathbb{R}_+^2 \). To be precise, for each household, consider the set \( C^{th} \) of consumptions \( c^{th} \) in \( \mathbb{R}_+^2 \) that are individually rational, \( u^{th}(c^{th}) \geq u^{th}(e^{th}) \); and feasible, \( e^{th} \leq \sum e^t_{-1,a} + e^t_{a+1,a} + e^t_{a+1,a} \). Call \( C^{th} \) the household’s rational-feasible set, and call the economy &E sufficiently curved if,

\footnote{An example in Okuno–Zilcha [21] shows that some such curvature restriction is required for the forthcoming price characterization of optimality.}
across all households and all rational feasible sets

\[
\inf_{t,h} \inf_{c_{t,h} \in \mathcal{C}_{t,h}} G^h(c^h) > 0
\]

(Note: Since there is no upper bound on curvature, the Gaussian curvature of some indifference curves can be \(+\infty\) (indifference curves can have kinks), and so utility functions need not be differentiable. Thus, even the original Cass–Okuno–Zilcha counter-examples meet that condition.)

We adopt sufficient curvature as a weak, maintained assumption, interpreted as goods across the two periods of anyone’s life are imperfect substitutes. For example, consider the economy with a constant, positive endowment for all households, and a single household in each generation, with constant-elasticity-of-substitution utility \(u'(c_t, c_{t+1}) = (1/\sigma_t)c_t^{1-\sigma_t} + (\beta/\sigma_t) c_{t+1}^{1-\sigma_t}\), for \(0 < \beta < +\infty\) and \(\sigma_t > 0\). Then sufficient curvature is satisfied if, and only if, the elasticities of substitution \(1/\sigma_t\) are bounded away from \(+\infty\) (the \(\sigma_t\) are bounded away from 0).

**Price-characterization Lemma.** For any sufficiently-curved economy \(\mathcal{E}\), \(^5\) consider any equilibrium \((d_1, d_2, \ldots)\) for which discount rates

\[
0 < \inf_{t} d_t \leq \sup_{t} d_t < +\infty
\]  

(3)

and the associated equilibrium consumptions

\[
0 < \inf_{t,h} c_{t,h} \leq \sup_{t,h} c_{t,h} < +\infty; \quad 0 < \inf_{t,h} c_{t+1,h} \leq \sup_{t,h} c_{t+1,h} < +\infty
\]  

(4), (5)

are bounded from below and from above. Then, the equilibrium is Pareto optimal if its discount factors are small enough to satisfy \(\sum_{t=1}^{\infty} 1/(d_1 \cdots d_t) = \infty\).

Looking ahead to our existence proof, we will only apply that price characterization to economies and equilibrium for which sufficient-curvature implies the boundedness conditions (3)–(5) for the lemma. (It also turns out that we will generate equilibrium discount factors that satisfy a stricter sense of smallness, that \(\sup_t(d_1 \cdots d_t) < +\infty\). Hence, one could generalize our results by rewriting a price-characterization lemma for that stricter sense of smallness; in particular, such a result could be proved with a weaker version of “sufficient curvature”.)

4. The generic existence of an optimal equilibrium: statement

The generic-existence result sets up with a short list of technical definitions.

**Continuous economies:** Let \(\mathcal{E}^{\infty}\) be the set of sufficiently-curved economies. Since utility functions and excess demands are continuous, but not necessarily differenti-

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\(^5\) Extensions of that price characterization, allowing more than one good in each period or two periods in each lifetime, require further bounds on the curvature of indifference curves beyond a lower bound on Gaussian curvature [9, p. 214]. Warning: Some price-characterization results in the literature are false as stated because they impose only a lower bound on Gaussian curvature for a many-good or many-lifetime economy.
able, call $E^0$ the set of continuous economies. Make $E^0$ into a topological space by defining the convergence $\mathcal{E}^k \to \mathcal{E}$ of economies as the uniform convergence

$$\sup_{ik} \| e^{ik}_k - e^{ik} \| \to 0$$

of endowments plus the uniform convergence

$$\sup_{i,k} \sup_{x \in e^k C} \| u^{ik}(x) - u^k(x) \| \to 0$$

of utility functions on each compact subset $C$ of $\mathbb{R}_+^2$.

Continuous-stationary economies: Let $S^0$ be the sub-space of $E^0$ of stationary economies; call them the continuous-stationary economies.

Smooth-stationary economies: For any $H$-list $u = (u^1, \ldots, u^H)$ of household utility functions that, when restricted to $\mathbb{R}_+^{2H}$, are continuously twice-differentiable and generate indifference curves with non-zero Gaussian curvature, let $S^2(u)$ be the space of stationary economies with fixed preferences $u = (u^1, \ldots, u^H)$, but alternative endowments $e = (e^1, \ldots, e^H)$ in $\mathbb{R}_+^2 \times \cdots \times \mathbb{R}_+^2 \times \mathbb{R}_+^{2H}$. Since it is parameterized by $\mathbb{R}_+^{2H}$, $S^2(u)$ is naturally endowed with the Euclidean topology and Lebesgue measure. Call them the smooth-stationary economies. Note: Every smooth-stationary economy is sufficiently curved, $S^2(u) \subset E^0$.

Robust optimality: An equilibrium $(d_1, d_2, \ldots)$ of an economy $\mathcal{E}$ in $E^0$ is robustly optimal if the equilibrium is Pareto optimal and if, for any sequence of economies $\mathcal{E}^k \to \mathcal{E}$ in $E^0$, $\mathcal{E}^k$ has a Pareto-optimal equilibrium $(d_1^k, d_2^k, \ldots)$ for each $k$ sufficiently large, and that sequence of equilibria uniformly converges in time, $\sup_{i} |d_i^k - d_i| \to 0$ as $k \to \infty$.

Proposition: The subset of stationary economies that have at least one robustly-optimal equilibrium is open-dense in the space $S^0$ of continuous-stationary economies, and is open-dense, full-measure in each spaces $S^2(u)$ of smooth stationary economies.

Simply put: An optimal equilibrium generically exists, and is robust, for nearly-stationary economies.

5. The generic existence of an optimal equilibrium: proof

By the definition of robust optimality, the set of economies with at least one robustly-optimal equilibria is open, relative to either $S^0$ or $S^2(u)$.

To prove density and full-measure, consider the class of stationary economies satisfying these regularity conditions:

R.1: $y: \mathbb{R}_+^{\infty} \to \mathbb{R}$ and $z: \mathbb{R}_+^{\infty} \to \mathbb{R}$ are continuously differentiable.
R.2: $y'(d) \neq 0$ whenever $y(d) = 0$.
R.3: $y(1) \neq 0$ and $y'(1) \neq z'(1)$.

Using some standard results of Debreu [13, 14], the subset of economies satisfying (R.1) and (R.2) is dense, full-measure in each space $S^2(u)$ of smooth stationary economies. And the regularity analysis of Kehoe and Levine, which allows many types of commodities in each period, can be specialized to prove that
the subset of economies satisfying (R.3) is dense in a space of smooth-stationary economies parameterized by excess-demand functions. While Kehoe and Levine’s analysis clearly has implications for the space considered in this paper, one need not bother translating their proof. For it’s easy enough to prove from scratch that the subset of economies satisfying (R.3) is dense, full-measure in each space \( S^d(\mu) \) of smooth stationary economies.

Finally, since the space of smooth-stationary economies is a dense subset of the space \( S^d \) of continuous-stationary economies [14, 18], it also follows that the subset of economies satisfying (R.1), (R.2), (R.3) is dense in the space \( S^d \) of continuous-stationary economies.

Putting it together, the proof is over once we show that every stationary economy \( \mathcal{E} \) satisfying the regularity conditions (R.1), (R.2), (R.3) has at least one robustly-optimal equilibrium.

Divide that proof into the following three cases.

Case 1: \( y(1) > 0 \), where discount factors are solved independently.

As asserted in the Introduction, the equilibrium discount factors for the economies can be found by either forward or backward substitution. But that is not the simplest proof in this case; here equilibrium discount factors can be solved independently.

To find the robustly-optimal equilibrium \((d, d, \ldots)\) for \( \mathcal{E} \), note from the monotonicity and continuity of utility that first-period consumption gets small as it gets expensive, \( \lim \sup_{d \to 0} y(d) < 0 \). And since we also have the reverse inequality when first-period consumption is cheap, \( y(1) > 0 \), the Intermediate Value Theorem implies zero excess demand, \( y(d) = 0 \), is achieved at some discount factor \( d \) between 0 and 1.

Hence, \((d, d, \ldots)\) is an equilibrium for \( \mathcal{E} \). And it’s Pareto optimal since \( d \leq 1 \) and \( \mathcal{E} \) is sufficiently curved.

To show that \((d, d, \ldots)\) is robustly-optimal note from (R.2) that \( y'(d) \neq 0 \) since \( y(d) = 0 \). Hence, for some compact sub-interval \( D \) of \((0, 1)\) containing \( d \) in its interior,

\[
\min_{d \in D} y(d_0) < 0; \quad \max_{d \in D} y(d_0) > 0
\]

Now consider any convergent sequence \( \mathcal{E}^k \to \mathcal{E} \) of continuous economies. The continuity and strict quasi-concavity of utility functions in the original economy \( \mathcal{E} \) and the uniform convergence \( u^{k} \to u^{\mathcal{E}} \) of preferences imply the convergence of excess-demand functions that is uniform in time and pointwise in the discount rate: for each discount rate \( d_0 \), \( \sup_t |y^k_t(d_0) - y(d_0)| \to 0 \). Now consider two points \( d_0, D_0 \in D \) described in the inequalities above, \( y(d_0) < 0 \) and \( y(D_0) > 0 \). Applying the uniform convergence (in time) of excess demand functions to those two points implies for each large \( k \), \( y^k_t(d_0) < 0 \) and \( y^k_t(D_0) > 0 \) for all \( t \). Thus:

\[
\min_{t \in D^k} y^k_t(d^k_t) < 0; \quad \max_{t \in D^k} y^k_t(d^k_t) > 0
\]

\[\text{In fact, in Case 3, we'll prove that the convergence of excess-demand functions is uniform in time and in any compact subset of discount rates.}\]
Hence, because each of those excess-demand functions are continuous, the Intermediate Value Theorem implies zero excess demand, \( y^t(d^t) = 0 \), is achieved at some discount factor \( d^t \) in \( D \).

Thus the sequence \((d^t_1, d^t_2, \ldots)\) of those discount factors is an equilibrium for \( \delta^t \). Since each discount factor was chosen from \( D \subseteq (0, 1) \), the equilibrium \((d^t_1, d^t_2, \ldots)\) satisfies the boundedness condition (3) from the price-characterization lemma and the smallness condition \( \sum_{i=1}^\infty 1/(d_1 \cdots d_i) = \infty \).

To proceed, reconsider the limit economy \( \delta \). Since, by assumption, each household prefers its endowment \( e^k \) to any consumption on the boundary of \( \mathbb{R}^2_+ \) and each utility function is continuous, the rational-feasible sets \( C^k \) for household types are each compact and are all contained in the interior of some compact subset \( C \) of \( \mathbb{R}^2_+ \). Hence, since preferences converge \( u^{h,k} \to u^k \) uniformly on that compact set \( C \), for large \( k \), the rational-feasible set \( C^{h,k} \) for each household in \( \delta^k \) is also contained in the interior of \( C \). In particular, since the consumption \( c^{h,k} \) in the equilibrium \( d^k \) for \( \delta^k \) are in the rational-feasible sets \( C^{h,k} \), the consumptions \( c^{h,k} \) are in the compact set \( C \subseteq \mathbb{R}^2_+ \) and, therefore, satisfy the boundedness conditions (4), (5) from the price-characterization lemma.

**Conclusion:** The equilibrium \( d^k \) is Pareto optimal.

Summing up so far, for each compact interval \( D \) containing \( d \) in its interior, \( \delta^k \) has a Pareto-optimal equilibrium \((d^t_1, d^t_2, \ldots)\) for each \( k \) sufficiently large, and each of those equilibrium discount rates is in \( D \). That is almost the statement that the equilibrium \((d, d, \ldots)\) of \( \delta \) is robustly optimal, but not quite. Finish the proof with a diagonal argument:

For \( n = 1, 2, \ldots \), for each compact interval \( D(n) = [d - \frac{1}{n}, d + \frac{1}{n}] \) centered on the steady state discount rate \( d \), the economy \( \delta^k \) has a Pareto-optimal equilibrium \((d^t_1(n), d^t_2(n), \ldots)\) for each \( k \geq K(n) \), for some sufficiently large constant \( K(n) \). And those equilibria satisfy \( \sup_i |d^t_i(n) - d| \leq \frac{1}{n} \).

Increase those constants, if necessary, so that \( K(1) < K(2) < \ldots \). For each \( k \) between \( K(n) \) and \( K(n + 1) - 1 \), set \((d^t_1, d^t_2, \ldots)\) equal to the equilibrium \((d^t_1(n), d^t_2(n), \ldots)\) described above. Piecing all those equilibria into a sequence \((d^t_1, d^t_2, \ldots)^{\infty}_{k=K(1)} \) we have, for each \( k \geq K(1), (d^t_1, d^t_2, \ldots) \) is a Pareto-optimal equilibrium of the economy \( \delta^k \), and that sequence of equilibria uniformly converges (in time) to \((d, d, \ldots)\), \( \sup_i |d^t_i - d| \to 0 \) as \( k \to \infty \). (For \( k \geq K(n) \), \( \sup_i |d^t_i - d| \leq \frac{1}{n} \). Thus, the equilibrium \((d, d, \ldots)\) of \( \delta \) is robustly optimal.

**Case 2:** \( y(1) < 0 \) and \( |y'(1)| > |z'(1)| \), where discount factors are solved by forward substitution.

The robustly-optimal equilibrium for \( \delta \) is to be unit sequence, \((1, 1, \ldots)\). Walras's law proves that it is an equilibrium, and it is Pareto optimal since the economy \( \delta \) is sufficiently curved.

To show that \((1, 1, \ldots)\) is robustly-optimal, note from the continuity of excess demand functions that the conditions \( y(1) < 0 \), \( |y'(1)| > |z'(1)| \) extend from the unit factor to some compact sub-interval \( D \) of \((0, + \infty) \) containing 1 in its interior:

\[
\max_{d \in D} y(d) < 0; \quad \min_{d \in D} |y'(d)| > \max_{d \in D} |z'(d-1)|
\]  

(6)
After restricting $D$ to be symmetric about 1, young-age excess demand being more sensitive than old-age excess demand (7), and Walras's law, $z(1) + y(1) = 0$, imply that any move in old-age excess demand can be offset by some move in young-age excess demand: no matter how high old-age demand can be, some young-age demand is low enough to make total excess demand negative

$$\max_{d_{-1} \in D} z(d_{-1}) + \min_{d_{<} \in D} y(d_{>) < 0$$ \hspace{1cm} (8)

and no matter how low old-age demand can be, some young-age demand is high enough to make total excess demand positive

$$\min_{d_{-1} \in D} z(d_{-1}) + \max_{d_{<} \in D} y(d_{>) > 0$$ \hspace{1cm} (9)

Now consider any convergent sequence $d^k \to d^*$ of continuous economies. As before, excess-demand functions converge uniformly in time and pointwise in the discount rate. In fact, excess-demand functions converge uniformly in time and uniformly in the set $D$ of discount rates: $\sup_{t} \sup_{d \in D} |z^k_t(d_{-1}) - y(d_{>)| \to 0$; likewise for old-age demand.

To prove that convergence, the stationarity of the limit economy $d^*$ and the uniform convergence (in time) of utility functions and endowments imply that it's sufficient to prove that each generation's excess-demand converges uniformly in the discount rate: for $t$ fixed, $\sup_{d \in D} |z^k_t(d_{-1}) - y(d_{>)| \to 0$; likewise for old-age excess-demand. And for that, since endowments converge, it's sufficient to prove that each individual household's total demand converges uniformly in the discount rate: for $t, h$ fixed, $\sup_{d \in D} |c^{h,k}_h(d_{-1}) - c^h(d_{-1})| \to 0$. To that end, assume otherwise for some fixed household $t, h$. Then, for some positive tolerance $\varepsilon$ and for an infinite number of $k$ (without loss of generality, all $k$),

$$|c^{h,k}(d^k) - c^h(d^k)| \geq \varepsilon$$

for some $d^k$ in $D$. Since $D$ is compact, some sub-sequence (without loss of generality, the entire sequence) of discount rates converges, $d^k \to d$. Hence, the budget constraints (1), $c^{h,k}(d^k) \leq (1, d^k) \bullet c^{h,k}(d^k)$ met by each demand $c^{h,k}(d^k)$ and the convergence of endowments imply that sequence of demand is bounded; hence, some sub-sequence (without loss of generality, the entire sequence) converges,

$$c^{h,k}(d^k) \to c^h$$ \hspace{1cm} (10)

Finally, since demand is obviously continuous, $c^h(d^k) \to c^h(d)$. Therefore, $|c^h - c^h(d)| \geq \varepsilon$, and $c^h \neq c^h(d)$.

But $c^h \neq c^h(d)$ leads to a contradiction. First, the convergence of consumptions (10), the convergence of endowments and discount rates, and the budget constraints met by each demand $c^{h,k}(d^k)$ imply that limit consumption $c^h$ meets the limit

---

7 The notation should be clear: $c^{h,k}(d_{-1}) \in R^k_+$ is the total demand (excess-demand plus endowment) by household $th$ in the non-stationary economy $d^k$ when the discount rate is $d^k$. $c^h(d_{-1})$, by household-type $h$ in the stationary economy $d^*$.
budget constraint, \((1, d) \cdot \tilde{c} \leq (1, d) \cdot e^h\). But since \(e^h \neq c^h(d)\), the strict quasi-concavity of preferences implies that something better, \(u^h(\tilde{c}) > u^h(c^h)\), also meets the budget constraint, \((1, d) \cdot \tilde{c} \leq (1, d) \cdot e^h\). And since wealth is positive \(((1, d) \cdot e^h > 0)\) and utility is continuous we may perturb that better thing \(\tilde{c}\) to satisfy \(u^h(\tilde{c}) > u^h(c^h)\) and \((1, d) \cdot \tilde{c} < (1, d) \cdot e^h\). Hence, the convergence of utility functions uniformly over the compact set consisting of \(\tilde{c}, e^h\) and every \(e^{h,k}(d^k)\), and the convergence of consumption \((10)\) implies \(u^{h,k}(\tilde{c}) > u^{h,k}(e^{h,k}(d^k))\), for large \(k\). But the convergence of discount rates and endowments implies \((1, d^k) \cdot \tilde{c} < (1, d^k) \cdot e^{h,k}\), for large \(k\). And those inequalities contradict the utility maximization from demand \(e^{h,k}(d^k)\). Q.E.D.

Applying that uniform convergence (in time and the set \(D\) of discount rates) of excess-demand, for each large \(k\), the inequalities \((6), (8), (9)\) for \(d^k\) are preserved by each generation \(t\) in the economy \(d^k\):

\[
\sup_{d^k \in D} \max_{d^k \in D} y^d_t(d^k) < 0
\]

\[
\max_{d^k \in D} z^t_{t-1}(d^k_{t-1}) + \min_{d^k \in D} y^d_t(d^k) < 0;
\min_{d^k \in D} z^t_{t-1}(d^k_{t-1}) + \max_{d^k \in D} y^d_t(d^k) > 0
\]

The discount factors for the Pareto-optimal equilibrium for \(d^k\) are taken from the set \(D\). Take the first factor \(d^k_1\) to be anything in \(D\). Because young-age excess demand is negative \((11)\), the first period excess demand inequality holds \((1)\). Working forward, \(d^k_1 \in D\), the continuity of the young-age excess demand \(y^d_2\), and the property that change in old-age excess demand \(z^t_1\) can be countered by a suitable change in young-age excess demand \((12)\)

\[
z^t_1(d^k_1) + \min_{d^k_2 \in D} y^d_2(d^k_2) < 0;
\]

\[
z^t_1(d^k_1) + \max_{d^k_2 \in D} y^d_2(d^k_2) > 0
\]

imply, by the Intermediate Value Theorem, the second period market-clearing equality \((2)\), \(z^t_1(d^k_1) + y^d_2(d^k_2) = 0\), holds for some discount factor \(d^k_2\) in \(D\). Continuing that process, from \(d^k_2\) find a \(d^k_3\) that satisfies third-period market clearing; and so on, filling out an equilibrium vector of discount factors for \(d^k\), with every factor in \(D\).

By its construction, the equilibrium \((d^k_1, d^k_2, \ldots)\) satisfies the boundedness condition \((3)\) from the price-characterization lemma. To see that it also satisfies the smallness condition, \(\sum_{t=1}^{\infty} 1/(d^k_1 \cdots d^k_t) = \infty\), note: The sequence

\[
y^d_1(d^k_1) + d^k_t z^t_1(d^k_1) = 0, y^d_2(d^k_2) + d^k_t z^t_2(d^k_2) = 0, y^d_3(d^k_3) + d^k_t z^t_3(d^k_3) = 0, \ldots
\]

of Walras's law equations and the sequence

\[
z^t_1(d^k_1) + y^d_2(d^k_2) = 0, z^t_2(d^k_2) + y^d_3(d^k_3) = 0, z^t_3(d^k_3) + y^d_4(d^k_4) = 0, \ldots
\]

of market-clearing equations imply that the sequence

\[
y^d_1(d^k_1), d^k_1 y^d_2(d^k_2), d^k_1 d^k_2 y^d_3(d^k_3), d^k_1 d^k_2 d^k_3 y^d_4(d^k_4), \ldots
\]

of discounted generational savings is constant. Hence, the bounded negativity of young-age excess demands \((11)\) implies that the sequence

\[
d^k_1, d^k_1 d^k_2, d^k_1 d^k_2 d^k_3, \ldots
\]
of compounding discount factors is bounded from above, by \( y^*_1(d^*_1) \) over \( \sup \text{sup}_{d^*_1 \in D} y^*_1(d^*_1) \). In particular, \( \sum_{t=1}^{\infty} 1/(d^*_1 \ldots d^*_t) = \infty \).

Hence, the proof ends as in the first case verbatim if we replace \( d^* \) with \( 1 \): First, keeping the interval \( D \) of discount rates fixed, the consumptions in the equilibrium \( d^* \) for \( \delta^* \) satisfy the boundedness conditions (4), (5) from the price-characterization lemma and so the equilibrium \( d^* \) is Pareto optimal. Then, looking across ever shorter intervals \( D(n) = [1 - \frac{1}{n}, 1 + \frac{1}{n}] \) and the associated sequences of equilibria \( (d^*_1(n), d^*_2(n), \ldots)_{n \rightarrow \infty} \), a diagonal argument sorts through such equilibria to find a single sequence of equilibria \( (d^*_1, d^*_2, \ldots)_{k \rightarrow \infty} \) that uniformly converges (in time) to \( (1, 1, \ldots) \), \( \sup_{t} |d^*_t - 1| \rightarrow 0 \) as \( k \rightarrow \infty \). Thus the equilibrium \( (1, 1, \ldots) \) of \( \delta^* \) is robustly optimal.

Case 3: \( y(1) < 0 \) and \( |y'(1)| < |z'(1)| \), where discount factors are solved by backward substitution.

First, we show that this is the last case. Under regularity (R.1) – (R.3), we are only left to check that if \( y(1) < 0 \), then \( y'(1) \neq -z'(1) \). To that end, differentiate Walras's law, \( y(d) + dz(d) = 0 \), at the unit factor to find \( y'(1) + z'(1) + z(1) = 0 \). Hence, Walras's law and \( y(1) < 0 \) imply \( y'(1) + z'(1) < 0 \), which implies \( y'(1) \neq -z'(1) \).

Finally, as Case 2, the robustly-optimal equilibrium for \( \delta^* \) is to be the unit sequence, \( (1, 1, \ldots) \). Walras's law proves that it is an equilibrium, and it is Pareto optimal since the economy \( \delta^* \) is sufficiently curved. In fact, even the proof that \( (1, 1, \ldots) \) is robustly-optimal in this case is nearly a copy of the proof in the last case, up to the point where the discount factors are found. Here are the preliminary results that carry over:

Consider any convergent sequence \( \delta^*_k \rightarrow \delta^* \) of economies. For every sufficiently-small compact sub-interval \( D \) of \( (0, + \infty) \) that contains 1 in its interior, fix any economy \( \delta^*_k \) for sufficiently large \( k \). Then, young-age excess demand is boundedly negative

\[
\sup_{t} \max_{d^*_t \in D} y^*_t(d^*_t) < 0
\]

And, for any discount factor \( d^*_{t+1} \) in \( D \), the change in young-age excess demand, from \( y(t) \) to \( y^*_t+1(d^*_{t+1}) \), can be countered by a suitable change in old-age excess demand

\[
\min_{d^*_t \in D} \frac{y^*_t+1(d^*_t)}{y^*_t(d^*_t)} + \frac{y^*_t(d^*_t)}{y^*_t+1(d^*_t)} < 0; \quad \max_{d^*_t \in D} \frac{y^*_t+1(d^*_t)}{y^*_t(d^*_t)} + \frac{y^*_t(d^*_t)}{y^*_t+1(d^*_t)} > 0
\]

Hence, the Intermediate Value Theorem implies that there exists some discount factor \( d^*_t \) in \( D \) satisfying market clearing

\[
z^*_t+1(d^*_t) + y^*_t+1(d^*_t) = 0
\]

for good \( t + 1 \).

Hence, the proof is more complicated than in Case 2; we must now find the equilibrium discount factors for the economy \( \delta^* \) by backward substitution — on a dynamical system that has no last period. To that end, consider the truncation of the economy at any time \( T \).
Fixing any \( d_{T+1}^{k,T} \) in \( D \) for the discount rate at time \( T+1 \), we know there exists some \( d_T^{k,T} \) in \( D \) for the discount rate at time \( T \) that satisfies market clearing (14) for good \( T+1 \). And from that \( d_T^{k,T} \) in \( D \), there exists some \( d_T^{k,T-1} \) in \( D \) for the discount rate at time \( T-1 \) that satisfies market clearing for good \( T \). And so on until we have some \( d_T^{k,T} \) in \( D \) for the discount rate at time \( 1 \) that satisfies market clearing for good 2. In summary, any sequence extension \((d_1^{k,T}, \ldots, d_T^{k,T}, \ldots)\) of that finite list of discount factors ensures market clearing for all goods \( 2, \ldots, T \). Note also that the market-clearing inequality for good 1 also holds (13).

Still keeping the economy \( \theta^k \) fixed, but passing to larger truncation times \( T \) ad infinitum, consider the sequence of sequence extensions \((d_1^{k,T}, \ldots, d_T^{k,T}, \ldots)\), each lying in \((D \times D \times \ldots)\). Since \((D \times D \times \ldots)\) is compact for the topology of pointwise convergence, there is a sub-sequence of sequence extensions that pointwise converges to some limit \((d_1^*, d_2^*, \ldots)\) in \((D \times D \times \ldots)\) – for each \( t \), \( d_t^* \rightarrow d_t^k \).

Since excess-demand functions are continuous, that limit sequence \((d_1^*, d_2^*, \ldots)\) is an equilibrium for the economy \( \theta^k \). As before, by its construction, the equilibrium \((d_1^*, d_2^*, \ldots)\) satisfies the boundedness condition (3) from the price-characterization lemma. And the sequence of Walras's law and market-clearing equations combines with the bounded negativity of young-age excess demands (13) to prove that the sequence

\[
d_1^k, d_1^* d_2^*, d_1^k d_2^k d_3^*, \ldots
\]

of compounding discount factors is bounded from above and so also satisfies the smallness condition, \( \sum_{t=1}^{\infty} 1/(d_1^k \cdots d_T^k) = \infty \).

Hence, the proof ends as in the first case verbatim if we replace \( d \) with 1: First, keeping the interval \( D \) of discount rates fixed, the consumptions in the equilibrium \( d^k \) for \( \theta^k \) satisfy the boundedness conditions (4), (5) from the price-characterization lemma and so the equilibrium \( d^k \) is Pareto optimal. Then, looking across shorter intervals \( D(n) = [1 - \frac{1}{n}, 1 + \frac{1}{n}] \) and the associated sequences of equilibrium \( (d_1^*(n), d_2^*(n), \ldots)_{k=n}^{\infty} \), a diagonal argument sorts through such equilibria to find a single sequence of equilibria \( (d_1^*(n), d_2^*(n), \ldots)_{k=n}^{\infty} \) that uniformly converges (in time) to \((1, 1, \ldots)\), \( \sup_n |d_t^k - 1| \rightarrow 0 \) as \( k \rightarrow \infty \). Thus the equilibrium \((1, 1, \ldots)\) of \( \theta^k \) is robustly optimal.

References

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