JOURNAL OF ECONOMIC THEORY **70**, 364–390 (1996)
ARTICLE NO. 0094

Equilibrium for Overlapping Generations in Continuous Time

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Received March 30, 1995; revised September 23, 1995

We prove the existence of general equilibrium for continuous-time overlapping-generations models. Previous theorems exclude all non-linear C.E.S. and von Neumann–Morgenstern preferences and exclude production. Our primitive assumptions are satisfied by such preferences and by all Markovian production technologies satisfying Bewley's assumptions for Arrow–Debreu models provided that storage is possible, at some finite rate of depreciation and some positive capacity. A non-existence example shows our Markovian and storage assumptions cannot be dropped. *Journal of Economic Literature* Classification Numbers: C62, D51, D90.

1. INTRODUCTION

We prove the existence of general equilibrium for continuous-time overlapping-generations models. Previous overlapping-generations theorems either assume discrete time [8] or exclude production and exclude all non-linear C.E.S. and von Neumann–Morgenstern preferences [1, 2, 7].[1] Because of the robustness of theorems spanning discrete and continuous time, the convenience of continuous time for computing closed-form solutions [4, Section 4.3], and the prominence of C.E.S. and von Neumann–Morgenstern preferences, we generalize the literature. Our assumptions are like Bewley's for continuous-time Arrow–Debreu models, which feature a finite population of finite-lived consumers or infinite-lived family dynasties. Except all non-linear C.E.S. preferences of consumers over finite lifetimes satisfy all our primitive assumptions while Bewley excludes all C.E.S.

* Some of the ideas in this paper were presented and improved at C.O.R.E. (Belgium) in the summer of 1992. I thank Max Stinchcombe for expositional advice.

[1] Precisely, the continuous-time literature requires lower semi-continuous preferences for the weak topology [1, Property 3, p. 276], which excludes all non-linear C.E.S. and von Neumann–Morgenstern preferences on atomless L_p spaces [10, p. 1845].

0022-0531/96 $18.00

preferences with elasticity of intertemporal substitution less than 1,[2] which are standard preferences macroeconomists fit to intertemporal data [9]. And we add assumptions that production technology be Markovian, with capital stocks summarizing the effect of past production on current possibilities; that preferences be proper, in a weak sense satisfied by all non-linear C.E.S. preferences; and that storage be possible, at some finite rate of depreciation and some positive capacity.

As in the Arrow–Debreu literature [3, Theorem 1 proof], our existence proof extracts an equilibrium candidate from the equilibria of a net of discrete-time subeconomies.[3] The key difference appears when proving the candidate preserves budget constraints from the subequilibria, which includes proving the consumption-price value map, $(x, p) \mapsto \int p(\tau)\, x(\tau)$, is (jointly) continuous for suitable topologies on subequilibrium consumption flows and price lists. The Arrow–Debreu literature proves continuity with Walras's law, computed by setting to zero the difference of the value of total demand and total supply. Subsequent literature extends Walras's law to the subset of pure-exchange overlapping generations in which a finite set of consumers own a portion of all wealth [1, 2, 7, 14], such as when land owned by the initial generation does not depreciate but grows constant dividends worth a fixed proportion of the endowments of subsequent generations. Our continuity proof for general overlapping generations in continuous time with production does not use Walras's law and allows storage to depreciate, provided only that depreciation rates are bounded from above for all storage up to some positive capacity, like chocolate with 1-second half-life and 1-gram capacity.

Unlike the Arrow–Debreu existence literature, the storage assumption we use to prove value continuity does not bound the maximum intensity of consumption flows a priori. Arrow–Debreu proofs find candidate consumption for continuous-time equilibrium as limits of subequilibrium consumptions. Suitable limits exist under the assumption in the literature that all feasible consumption is in a weakly compact set. And such a compact set bounds flows a priori, excluding some flows like x^n that indicate 1 unit of stock is extracted and consumed at uniform rate n over the time interval $[1 - 1/n, 1]$ because such flows have no weak limit in the commodity space, as $n \to \infty$. Our accommodation of such flows, therefore, requires alternative candidates for equilibrium consumption. To introduce those candidates, consider part of an economy containing a single good storable over a bounded interval $[0, 1]$ of time. With no a priori bound on storage

[2] Precisely, Bewley requires Mackey lower semi-continuous preferences, which excludes elasticity less than 1.

[3] Bewley [3] first uses discrete approximations for continuous-time Arrow–Debreu economies. Mas–Colell and Zame [10] summarizes subsequent uses and alternative proofs.

flows, the natural commodity space is $L_1 = L_1[0,1]$, with price space $L_\infty = L_\infty[0,1]$. (Modeling behavior over an overlapping sequence of such intervals forms overlapping-generations economies.)

We adapt to the lack of subequilibrium limits in L_1 by finding limits in the larger set L'_∞, the norm dual of L_∞. According to the Alaoglu theorem, weak limits exist in L'_∞ under our assumption that all feasible consumption is in an L_1-bounded set. According to the Yosida–Hewitt theorem, each such limit is the sum of a consumption flow plus consumption spikes. Removing the spikes yields our candidate consumption flow in L_1. For instance, the consumption sequence $\{x^n\}$ above converges to a spike, with 1 unit of stock instantaneously extracted and consumed at time 1, which yields zero as the candidate flow. To prove utility maximization, utility from candidate consumption must equal the limit of utilities from subequilibria. To that end, we assume *surge* properness, the standard topological properness assumption on preferences modified to limit the contribution of a surge of feasible consumption. In particular, surge properness implies the decreasing duration of consumption in the sequence $\{x^n\}$ outweighs the increasing intensity of consumption, making utility contributions vanish, $u(x^n) \to u(0)$. Our approach of extracting candidate consumption flows in L_1 from a limit in L'_∞ is thus like Bewley's approach [3] of extracting candidate price lists in L_1, except all non-linear C.E.S. preferences of consumers satisfy all our assumptions while Bewley excludes those with elasticity less than 1.

Finally, a non-existence example shows the assumptions of Markovian technology and storage cannot be dropped. In fact, not only does that non-existence example satisfy all our assumptions except Markovian technology and storage, but preferences are linear and so satisfy the additional assumptions in the overlapping-generations and Arrow–Debreu literature, and technology satisfies Zame's additional assumption of bounded marginal rates of transformation.

Here is the plan of the rest of the paper, and a review of the principal results.

Section 2 contains standard assumptions, definitions, and notation for continuous-time overlapping-generations economies and equilibria. Because of storage, the space for each commodity over each bounded interval of time is L_1, with price space L_∞. That reverses Bewley's well-known commodity–price pairing $\langle L_\infty, L_1 \rangle$. All assumptions are like Bewley's, except we drop Mackey lower semi-continuity of preferences.

Section 3 begins with definitions and notation for storage. We assume storage is possible, at some finite rate of depreciation and some positive capacity. Unlike the Arrow–Debreu literature, we impose no a priori bound on the maximum intensity of storage flows. The section further compares the storage assumption to the literature by showing it implies certain aggregate production sets, formed by adding individual production sets, have non-empty norm interiors and, therefore, are proper [13], for the norm topology. Finally, the section explains the complexity of our forthcoming existence proof by showing storage does not imply a non-empty (weak) $\sigma(L_1, L_\infty)$ interior; such an interior would yield a simpler proof of the joint-continuity of the value map, $(x, p) \mapsto \int p(\tau)\, x(\tau)$, over subequilibrium commodity flows and price lists.

Section 4 introduces the non-standard properness assumption and evaluates its strength by observing that property for a neoclassical economy with non-linear C.E.S. preferences and Markovian technology.

Section 5 proposes the existence of equilibrium under standard assumptions plus the non-standard assumptions of surge-proper preferences, Markovian technology, and storage. The section comments on extensions to ensure storage flows are bounded in the smaller commodity space $L_\infty \subset L_1$, and to continuous labor input that is not storable.

Section 6 begins the proof of the existence theorem like the Arrow–Debreu literature, approximating each continuous-time economy with a net of discrete-time subeconomies. The section extracts a candidate for continuous-time equilibrium consumption from a limit of subeconomy equilibria so that utility from the candidate equals the limit of utility from subequilibria. Section 7 finishes the proof, showing candidate consumption and production satisfy equilibrium budget constraints and profit maximization.

Finally, Section 8 contains the non-existence example. It remains to be seen whether the assumptions of our overlapping-generations existence theorem, which are interpretable and satisfied by prominent applications when commodities are differentiated by continuous time, adapt to other continuum models, like uncertainty, location, or commodity differentiation, or whether such models include robust non-existence examples satisfying all interpretable properties of prominent applications.

2. STANDARD ASSUMPTIONS, DEFINITIONS, AND NOTATION

This section contains standard assumptions, definitions, and notation for continuous-time overlapping-generations economies and equilibria. Because of storage, the space for each commodity over each bounded interval of time is L_1, with price space L_∞. That reverses Bewley's well-known commodity–price pairing $\langle L_\infty, L_1 \rangle$. All assumptions are like Bewley's, except we drop Mackey lower semi-continuity of preferences.

2.1. *Economies*

Time is continuous, with beginning but without end. Each consumer is born at a discrete instant of time, and lives during a bounded interval of time. During each bounded interval of time only a finite number of consumers are alive. Hence, calibrate time so that the oldest consumers are born during interval $[0, 1)$, with some (exogenously) bearing new consumers during interval $[1, 2)$ and all dying by time 2. Calibrating all descendants puts each (past, present, future) consumer living during some interval $[t, t+2)$, for $t=0, 1, ...$. To simplify notation, model all consumers as living at all time during their interval $[t, t+2)$. For instance, model a consumer that dies early as a living consumer that eventually has zero endowments and gets no utility from consumption. (Excluding time $t+2$ from the lifetime interval is for later convenience.) For future reference, partition time into intervals $[0, 1)$, $[1, 2)$, ..., calling $[t-1, t)$ period t, and partition consumers into generations according to lifespan, calling the consumers living during periods g and $g+1$ the generation g.

Looking across the infinite number of generations, there are a (countably) infinite number of consumers, $i=1, 2, ...$. During each period, there are available a finite number of goods, $n=1, ..., N$. (To reduce notation, the number of goods is constant across periods.) For τ in $[0, 1)$, let $x_i^{nt}(\tau)$ denote consumption of good n at time $t+\tau$ in period t by consumer i. Thus a function $x_i^{nt}=x_i^{nt}(\cdot)$ denotes a consumption stream of a particular good by a particular consumer during a particular period. Restrict such functions to the non-negative orthant L_{1+} of the classical Banach space $L_1 = L_1[0, 1)$, the set of (Lebesgue) integrable functions on $[0, 1)$. For future reference to consumption in generation g, form the vectors

$$x_i^t=(x_i^{1t}, ..., x_i^{Nt}) \in L_{1+}^N \qquad \text{and} \qquad x_i=(x_i^g, x_i^{g+1}) \in L_{1+}^{N2},$$

where L_{1+}^N is the N-fold Cartesian product of L_{1+}; $L_{1+}^{N2} := (L_{1+}^N)^2$, the 2-fold product of L_{1+}^N. Denote the lifetime consumption set of each consumer $X_i := L_{1+}^{N2}$, and of the population of all consumers $X := \prod_i X_i$, with typical element $x=(x_i) \in X$.

Each consumer of generation g has a (strict) preference order $\succ_i$ over vectors in X_i, has a fixed endowment vector e_i in X_i, and has free access to a production possibility set Y_g, a subset of vectors $y_g=(y_g^g, y_g^{g+1})$ in L_1^{N2} defining net output during periods g and $g+1$. For future reference to production, form the product $Y := \prod_g Y_g$ of all (overlapping) production sets, with typical element $y=(y_g) \in Y$. Putting it all together, for fixed parameter N, an *economy* is a set $\mathscr{E}=\{X_i, \succ_i, e_i, Y_g\}$.

2.2. *Math Notation*

The product space L_1^{N2} inherits the following properties from Banach space $L_1 = L_1[0, 1)$.

1. *Order.* From the ordering of L_1, define two orders over vectors $z=(z^{nt})$ in L_1^{N2}:

- $z \geqslant 0$ if $z^{nt}(\tau) \geqslant 0$ for each $\tau \in [0, 1)$, each of the N goods n, and each of the 2 periods t.
- $z \gg 0$ if $\inf_{\tau, n, t} z^{nt}(\tau) > 0$.

2. *Norm.* From the norm on L_1, define the norm

$$\|z\| := \sum_n \sum_t \int_0^1 |z^{nt}(\tau)| \, d\tau$$

of vectors in L_1^{N2}.

The final properties involve elements $p=(p^{nt})$ of the 2-fold Cartesian product $L_\infty^{N2} := (L^N)^2$ of the N-fold Cartesian product of the classical Banach space $L_\infty = L_\infty[0, 1)$, the set of essentially bounded (Lebesgue) measurable functions on $[0, 1)$.

3. *Product.* From the definition of the product of elements of L_1 with elements of L_∞, define the product of $z=(z^{nt})$ in L_1^{N2} and $p=(p^{nt})$ in L_∞^{N2} by

$$p \cdot z := \sum_n \sum_t p^{nt} z^{nt}, \qquad \text{where} \qquad p^{nt} z^{nt} := \int_0^1 p^{nt}(\tau)\, z^{nt}(\tau)\, d\tau$$

4. *Topology.* From the weak and Mackey topologies over L_1 for dual L_∞, we have weak $\sigma = \sigma\langle L_1^{N2}, L_\infty^{N2}\rangle$ and Mackey $\tau = \tau\langle L_1^{N2}, L_\infty^{N2}\rangle$ topologies over the commodity space L_1^{N2}. In particular, a net $\{z^\alpha\}$ of vectors (weak) σ-converges to 0 if, and only if, $p \cdot z^\alpha \to 0$ for each p in L_∞^{N2}.

2.3. *Allocations*

An *allocation* is a consumption–production pair $(x, y) = ((x_i), (y_g))$ in $X \times Y$ that, during each time period t, balances commodity materials

$$\sum_i (x_i^t - e_i^t) = \sum_g y_g^t, \tag{1}$$

where the summations are restricted to consumers and generations active during period t. For instance, when $t > 1$, the consumers are those in generations $t-1$ and t.

For each consumer, define the set X_i^f of feasible consumptions as those consumptions in X_i that are part of some allocation. Likewise, for each generation, define the set Y_g^f of feasible productions.

2.4. *Standard Assumptions*

The following assumptions are like Bewley's [3], except we drop Mackey lower semi-continuity of preferences.

A.1. Each preference order $\succ_i$ is asymmetric, negatively transitive, and admits free disposal in the sense that no pair of consumptions satisfy both $x \succ_i \hat{x}$ and $\hat{x} \geqslant x$. And some consumption $\hat{x}$ in X_i bounds the order in the sense, for each consumption x in X_i, $\alpha\hat{x} \succ_i x$ for some $\alpha > 0$.

A.2. Each preference order $\succ_i$ is Mackey upper semi-continuous in the sense that each inferior set $\{\hat{x} \in X_i : x \succ_i \hat{x}\}$ is relatively (Mackey) τ-open in X_i. Each order is Euclidean lower semi-continuous in the sense that, for each finite-dimensional restriction $X_i \cap F$ of the commodity space, the superior set $\{\hat{x} \in (X_i \cap F) : \hat{x} \succ_i x\}$ is relatively open in the Euclidean topology.

A.3 Each preference order $\succ_i$ is convex in the sense that each superior set is convex, and the complement of each inferior set is convex.

A.4 Each production set Y_g is a convex, (weak) σ-closed cone at the origin. Production can always be truncated in progress; that is, the truncation $(y_g^g, 0) \in Y_g$ whenever $(y_g^g, y_g^{g+1}) \in Y_g$.

A.5. The feasible-consumption sets X_i^f and the feasible-production sets Y_g^f are norm-bounded subsets of L_1^{N2}.

A.6. For any allocation (x, y) of individually rational consumption $(e_i \not\succ_i x_i)$ and any partition $\{I, \hat{I}\}$ of the set of consumers, at least one consumer i in I desires the endowments of the consumers in $\hat{I}$; that is, $x_i + z \succ_i x_i$ for some $z \leqslant \sum_{\hat{\imath} \in \hat{I}} e_{\hat{\imath}}$.

The order (A.1) and continuity (A.2) assumptions imply that each preference order is represented by a utility function, u_i: $X_i \to \mathbb{R}$. In fact, the proof of Theorem 1 of Richard and Zame [15, p. 241] proves existence of utility using only our assumptions, without their stronger continuity and properness assumptions. (The existence of the described bound (A.1) is a weak assumption; it follows from our forthcoming surge-properness assumption).

Euclidean lower semi-continuity (A.2) is weaker than Mackey lower semi-continuity, which Bewley imposes. In particular, Cobb–Douglas preferences $u_i(x_i) = \sum_{nt} \int_0^1 \ln(x_i^{nt}(\tau))$ and C.E.S. preferences with elasticity of intertemporal substitution less than 1 are Euclidean lower semi-continuous, but not Mackey.

The cone (A.4) assumption on the production set can be dropped. As in Arrow–Debreu models [11, Section 5], having a fixed, finite collection of convex production sets owned by shareholders is equivalent to our choice of a single convex production cone freely available to all consumers within a generation, provided that our list of commodities include entrepreneurial inputs.[4]

2.5. *Equilibrium*

The following definition uses vectors in $L^N_\infty = L^N_\infty[0, 1)$ to discount all value within each time period to the beginning of the economy, at time 0. For instance, the time 0 value of a production stream $y_g = (y^g_g, y^{g+1}_g)$ in Y_g for generation g is the product

$$(p^g, p^{g+1}) \cdot y_g := \sum_n p^{ng} y^g_g + \sum_n p^{n,\, g+1} y^{g+1}_g$$

for a pair of vectors p^g and p^{g+1} in L^N_∞. Denote the price set for each period $P^t := L^N_\infty$, and across all periods $P := \prod_t P^t$.

An *equilibrium* is a price sequence $p = (p^t)$ in P such that some solution to the budget-constrained optimization problem

$$\begin{array}{ll} \text{Maximize} & u_i(x_i) \\ \text{over} & x_i \in X_i \\ \text{subject to} & (p^g, p^{g+1}) \cdot (x_i - e_i) \leqslant 0 \end{array}$$

of each consumer in each generation g matches with some solution to each generation's the profit-maximization problem

$$\begin{array}{ll} \text{Maximize} & (p^g, p^{g+1}) \cdot y_g \\ \text{over} & y_g \in Y_g \end{array}$$

to form an allocation (x, y).

3. STORAGE

This section begins with definitions and notation for storage. Consider storage of an initial stock $s_0 > 0$ of a single good over time in $[0, 1)$, where the stock is bounded by initial capacity $\bar{s} > 0$, and both the stock and capacity depreciate, at exponential rate δ. Formally, the storage set

[4] In our model, entrepreneurial inputs are included as "discrete" goods, as defined in Section 4.

$S = S(s_0, \bar{s}, \delta)$ consists of all net-output flows y in L_1 satisfying the storage inequalities

$$0 \leqslant s_0 - \int_0^t e^{\delta\tau} y(\tau)\, d\tau \leqslant \bar{s} \tag{2}$$

for each t in $[0, 1)$. To understand those inequalities, note a flow y in L_1 satisfies the storage inequalities if, and only if, the function

$$s(t) := s_0 e^{-\delta t} - e^{-\delta t} \int_0^t e^{\delta\tau} y(\tau)\, d\tau$$

is absolutely continuous, and is differentiable almost everywhere in $[0, 1)$, and when differentiable, satisfies the differential equation [16, Theorem 5.10, p. 107]

$$s'(t) = -y(t) - \delta s(t) \qquad \text{subject to} \qquad s(0) = s_0$$
$$\text{and} \qquad 0 \leqslant s(t) \leqslant \bar{s} e^{-\delta t}.$$

That is, $s(t)$ describes accumulated storage at time t, with the change $s'(t)$ in storage equal to net input $-y(t)$ minus depreciation equal to the fraction δ of existing storage.

Assume each good is storable, at some finite rate of depreciation and positive capacity.

A.7 Each good n is *storable* in the sense that, for each period t, for some storage set $S_t = S(s_0, \bar{s}, \delta)$, each element of S can be net-output during period t using consumer 1's endowment as input. That is, for period 1, for each s^n in S_1,

$$e_1^1 + \bar{y}_1^1 = (0, ..., s^n, 0, ...)$$

for some production $\bar{y}_1 = (\bar{y}_1^1, \bar{y}_1^2)$ in Y_1. For period 2, for each s^n in S_2,

$$e_1^1 + \bar{y}_1^1 = 0 \qquad \text{and} \qquad e_1^2 + \bar{y}_1^2 + \bar{y}_2^2 = (0, ..., s^n, 0, ...)$$

for some pair $\bar{y}_1$ in Y_1 and $\bar{y}_2$ in Y_2. And for each period $t > 2$, for each s^n in S_t,

$$e_1^1 + \bar{y}_1^1 = 0, \qquad e_1^2 + \bar{y}_1^2 + \bar{y}_2^2 = 0,$$
$$\bar{y}_2^3 + \bar{y}_3^3 = 0, ..., \qquad \bar{y}_{t-1}^t + \bar{y}_t^t = (0, ..., s^n, 0, ...)$$

for some list $\bar{y}_1$ in Y_1 through $\bar{y}_t$ in Y_t.

There, consumer 1 is some member of the first generation. The simplest example of storage (A.7) is the neoclassical economy in Section 4, where consumer 1's endowment includes a positive capital stock, which can be stored then released for future consumption.

Unlike the Arrow–Debreu literature, assumption A.7 imposes no a priori bound on the maximum intensity of storage flows, but the assumption is weak because depreciation can be big and capacity can be small. Thus chocolate is "storable" even if it had a half-life of 1 second and capacity of 1 gram.

We compare the storage assumption to the literature by showing it implies, for each (truncation) period T, the set $Y^T \subset L_1^{NT}$ of net-output production vectors over periods 1 through T that can be obtained from production by generations 1 through T has a non-empty norm interior and, therefore, is proper [13], for the norm topology. Because of standard assumptions plus storage (A.7), the non-empty interior for each truncated set Y^T follows from this observation about individual storage sets:

OBSERVATION 1. *Each storage set* $S = S(s_0, \bar{s}, \delta)$ *has a non-empty* $L_1[0, 1)$-*interior, which includes the origin.*

Proof of non-empty interior. The general observation follows from Lemma 3 in Section 7. But the proof for zero depreciation is obvious because, in that case, the storage inequalities (2) are satisfied by every vector in $L_1[0, 1)$ with norm less than both s_0 and $\bar{s}$. ▮

Finally, we explain the complexity of our forthcoming existence proof by showing storage does not imply a non-empty (weak) $\sigma(L_1, L_\infty)$ interior for any truncated set Y^T; such an interior would immediately prove joint-continuity of the value map, $(x, p) \mapsto \int p(\tau)\, x(\tau)$, over subequilibrium commodity flows and price lists.

OBSERVATION 2. *Each storage set* $S = S(s_0, \bar{s}, \delta)$ *has an empty (weak)* $\sigma\langle L_1, L_\infty\rangle$-*interior.*

Proof of empty interior. We show the origin is not in the interior by finding a net outside S that converges to the origin. Without loss of generality, choose a storage set with initial stock $s_0 < 1/2$. Define the flow $z^k := k\chi_{[0,\, 1/k)}$, which indicates 1 unit of stock is extracted and consumed at uniform rate k over the time interval $[0, 1/k)$. That sequence $\{z^k\}$ is norm-bounded in $L_\infty^{N\prime}$, the norm dual of L_∞^N. Hence, the Alaoglu theorem [17, page 84] implies some subnet $z^{k(\alpha)}$, $\alpha \in (A, \geqslant)$, (weak) $\sigma\langle L_\infty^{N\prime}, L_\infty^N\rangle$-converges to an element of $L_\infty^{N\prime}$. For each α, choose $\beta(\alpha) \geqslant \alpha$ such that $k(\beta(\alpha))/2 \geqslant k(\alpha)$. Hence, form the net

$$x^\alpha := z^{k(\beta(\alpha))} - z^{k(\alpha)}.$$

First, $\beta(\alpha) \geqslant \alpha$ implies the net $\{z^{k(\beta(\alpha))}\}$ convergence to the limit of $\{z^{k(\alpha)}\}$, meaning x^α (weak) $\sigma\langle L_1^N, L_\infty^N\rangle$-converges to the origin. But, $k(\beta(\alpha))/2 \geqslant k(\alpha)$ implies the flow

$$x^\alpha = z^{k(\beta(\alpha))} - z^{k(\alpha)} \notin S$$

The reason is that, for each time $t \leqslant 1/k(\beta(\alpha))$, $z^{k(\beta(\alpha))}$ specifies constant output $k(\beta(\alpha))$ while $-z^{k(\alpha)}$ specifies constant input $k(\alpha)$, making constant net-output flow $k(\beta(\alpha)) - k(\alpha)$, which accumulates to

$$\frac{k(\beta(\alpha)) - k(\alpha)}{k(\beta(\alpha))} \geqslant \frac{k(\beta(\alpha)) - k(\beta(\alpha))/2}{k(\beta(\alpha))} = \frac{1}{2}$$

at time $t = 1/k(\beta(\alpha))$, which is greater than the initial stock s_0.

To extend the proof to non-origin vectors z in L_1, again choose a storage set with initial stock $s_0 < 1/2$. Since the net $\{x^\alpha\}$ above converges to the origin, the net $z + x^\alpha$ converges to z. But for each time $\tau \leqslant 1/k(\beta(\alpha))$, the net-output flow $z(\tau) + k(\beta(\alpha)) - k(\alpha)$ accumulates to

$$\int_0^t z(\tau) + \frac{k(\beta(\alpha)) - k(\alpha)}{k(\beta(\alpha))} \geqslant \int_0^t z(\tau) + \frac{k(\beta(\alpha)) - k(\beta(\alpha))/2}{k(\beta(\alpha))} = \int_0^t z(\tau) + \frac{1}{2}$$

at time $t = 1/k(\beta(\alpha))$, which, for large α, is greater than the initial stock s_0, since $z \in L_1$ implies $\int_0^t z(\tau) \to 0$ as $t \to 0$. Hence, $z + x^\alpha \notin S$ for large α. ∎

4. NON-STANDARD ASSUMPTIONS

This section introduces the non-standard properness assumption and evaluates its strength by observing that property for a neoclassical economy with non-linear C.E.S. preferences and Markovian technology.

The neoclassical economy contains a single consumption good and a single capital good, used to produce the consumption good. Each consumer of each generation g has positive lifetime endowment (bounded away from the origin) of the consumption good, with preferences represented by non-linear C.E.S. utility

$$u_i(x_i) = \sum_n \int_0^1 \frac{1}{\rho} (x_i^{ng}(\tau))^\rho \, d\tau + \sum_n \int_0^1 \frac{1}{\rho} (x_i^{n, g+1}(\tau))^\rho \, d\tau$$

for $\rho < 1$ and $\rho \neq 0$. Consumer 1, the member of generation 1 mentioned in the storage assumption (A.7), also has a positive endowment of capital stock at time 0, which combines with an input flow of the consumption good to produce output according to a neoclassical growth equation, such as described in Burke [6]. Such economies typically satisfy standard assumptions (A.1 to A.6), plus storage (A.7). We shall prove two additional properties of the typical neoclassical economy. (We later impose those properties as non-standard assumptions on any economy.)

Formulating the additional properties involves trade in discrete time. There are many ways to model discrete time: restrict trade to the instant beginning each period, like firms trading capital stock; or restrict trade to constant functions within each period, like firms using an inelastically-supplied labor or entrepreneurial input; or restrict preferences and technology so that all equilibrium trades are equivalent to constant functions within each period. But an equilibrium under any one of those models yields an equilibrium under any other model. Hence, existence need only be proved for one model. We chose the first model to describe the neoclassical economy but now choose the last model to generalize the neoclassical properties since that model is consistent with previous definitions and assumptions for continuous time; in fact, that model makes discrete time a special case of continuous time.

Formally, good n is *discrete* in economy $\mathscr{E}$ if commodities in each period are perfect substitutes in both consumption and production. That is, for each consumer of each generation g and each consumption x_i in X_i, each other consumption $\hat{x}_i$ in X_i yields equal utility if the two consumptions agree for every good except n, and for good n

$$\int_0^1 [x_i^{ng}(\tau) - \hat{x}_i^{ng}(\tau)]\, d\tau = 0 \qquad \text{and} \qquad \int_0^1 [x_i^{n,\,g+1}(\tau) - \hat{x}_i^{n,\,g+1}(\tau)]\, d\tau = 0.$$

For each production y_g in Y_g, each other vector $\hat{y}_g$ in L_1^{N2} is also in Y_g if the two vectors agree for every good except n, and for good n

$$\int_0^1 [y_g^{ng}(\tau) - \hat{y}_g^{ng}(\tau)]\, d\tau = 0 \qquad \text{and} \qquad \int_0^1 [y_g^{n,\,g+1}(\tau) - \hat{y}_g^{n,\,g+1}(\tau)]\, d\tau = 0$$

Finally, call a good *continuous* if it is not discrete.

The first property of the neoclassical economy concerns the timing of inputs and outputs of continuous goods.

OBSERVATION 3. *Technology in the neoclassical economy is Markovian in the sense that no continuous good can be inputed or outputted after each generation's midlife; that is, for each net-output y_g in the production set Y_g, $y_g^{n,\,g+1} = 0$ for each continuous good n.*

The Markovian property follows whenever capital stocks summarize the effect of past production decisions on current and future production possibilities. That is, there is no loss of generality to suppose that a firm will only produce during its first period then sell its remaining capital stock to the generation born at the start of the next period. Thus, for a judicious choice of each generation's production set, only capital stocks, which are discrete goods, can be traded after the generation's midlife.

For the second property of the neoclassical economy, define notation that truncates consumption at selected times. For each consumption x in $L^{N2}_{1+}[0, 1)$ and each measurable subset E of $[0, 1)$, define truncated consumption x_E in L^{N2}_{1+} by

$$x^{nt}_E(\tau) := \begin{cases} 0 & \text{if good } n \text{ is continuous and } \tau \notin E \\ x^{nt}(\tau) & \text{otherwise.} \end{cases}$$

That is, x_E truncates consumption to 0 for continuous goods off the time interval E in each period.

OBSERVATION 4. *Each preference order in the neoclassical economy is surge proper in the sense, for each vector $z \gg 0$ in X_i, there exists a positive ε such that*

$$x_{iE} + z \succ_i x_i$$

for each x_i in X^f_i and each set $E \subset [0, 1)$ with (Lebesgue) measure $\mu(E) > 1 - \varepsilon$.

Surge properness is an original form of standard topological properness restrictions on preferences modified to limit the contribution of a surge of feasible consumption.

Proof of surge properness. To reduce notation, just consider the utility discount rate $\delta = 0$ and drop the consumer subscript. Fix any vector $z \gg 0$ in X_i.

Fix any positive $\varepsilon < 1/4$, and consider each x in X^f and each set $E \subset [0, 1)$ with measure $\mu(E) > 1 - \varepsilon$. The proof is to show that, when ε is small enough, $u(x_E + z) - u(x) > 0$. Hence, compute the utility difference

$$u(x_E + z) - u(x) = \sum_t \int_{[0,1)} \frac{1}{\rho} [(x^t_E(\tau) + z^t(\tau))^\rho - (x^t(\tau))^\rho]\, d\tau, \qquad (3)$$

where t is summed over periods $t = g$ and $t = g + 1$. Let b bound the norm of all feasible consumption (A.6). For each period t, let $B_t := \{\tau : x^t(\tau) \leqslant 2b\}$, and partition the domain $[0, 1)$ of integration into three sets—$(B_t \cap E)$, the complement E^c of E, and $(B^c_t \cap E)$. (For future reference, from the definition of b as a norm bound, $\mu(B_t) \geqslant 1/2$, which with $\mu(E) > 1 - \varepsilon > 3/4$ bounds the measure of the first two partition sets—$\mu(B_t \cap E) > 1/4$ and $\mu(E^c) < \varepsilon$.) Since the instantaneous utility difference

$$\frac{1}{\rho}[(x_E^t(\tau) + z^t(\tau))^\rho - (x^t(\tau))^\rho]$$

is non-negative over the third set $(B_t^c \cap E)$, the integrated difference (3) is positive provided

$$\int_{B_t \cap E} \frac{1}{\rho}[(x^t(\tau) + z^t(\tau))^\rho - (x^t(\tau))^\rho]\,d\tau + \int_{E^c} \frac{1}{\rho}[(z^t(\tau))^\rho - (x^t(\tau))^\rho]\,d\tau > 0 \tag{4}$$

in each period $t = g, g+1$. (Recall: $x_E^t = x^t$ over $B_t \cap E$ and $x_E^t = 0$ over E^c.)

To bound the first term $\int_{B_t \cap E} (1/p)[(x^t(\tau) + z^t(\tau))^\rho - (x^t(\tau))^\rho]\,d\tau$ from below, consider the instantaneous utility difference

$$\frac{1}{\rho}[(x^t(\tau) + z^t(\tau))^\rho - (x^t(\tau))^\rho]$$

for each τ in $B_t \cap E$. The definition of B_t implies $x^t(\tau) \leqslant 2b$, so concavity of the felicity function implies the utility contribution of adding $z^t(\tau)$ to $x^t(\tau)$ is at least as large as the utility contribution of adding to $2b$; that is,

$$\frac{1}{\rho}[(x^t(\tau) + z^t(\tau))^\rho - (x^t(\tau))^\rho] \geqslant \frac{1}{\rho}[(2b + z^t(\tau))^\rho - (2b)^\rho].$$

Hence, for $\underline{z}$ defined as the essential infimum $\min_t \operatorname{ess\,inf}_\tau z^t(\tau) > 0$ of positive flow $z \gg 0$, and for $\alpha := (1/p)[(2b + \underline{z})^\rho - (2b)^\rho] > 0$

$$\int_{B_t \cap E} \frac{1}{\rho}[(x^t(\tau) + z^t(\tau))^\rho - (x^t(\tau))^\rho]\,d\tau \geqslant \alpha\mu(B_t \cap E) > \alpha/4 \tag{5}$$

since $\mu(B_t \cap E) > 1/4$.

To bound the second term $\int_{E^c} (1/\rho)[(z^t(\tau))^\rho - (x^t(\tau))^\rho]\,d\tau$ from below, we bound utility

$$\int_{E^c} \frac{1}{\rho}(x^t(\tau))^\rho\,d\tau \tag{6}$$

from above. To that end, the feasibility of consumption x bounds its (norm) integral $\int_{E^c} x^t(\tau)\,d\tau \leqslant b$. Hence, concavity of the felicity function implies utility from consumption x at most equals utility from the flow that is constant over E^c in period t and has norm b; that flow equals $b/\mu(E^c)$ over E^c in period t. Precisely,

$$\int_{E^c} \frac{1}{\rho} (x^t(\tau))^\rho \, d\tau \leqslant \int_{E^c} \frac{1}{\rho} (b/\mu(E^c))^\rho \, d\tau$$
$$= \frac{1}{\rho} (b/\mu(E^c))^\rho \, \mu(E^c)$$
$$= \frac{b^\rho}{\rho} \mu(E^c)^{1-\rho}.$$

But $\mu(E^c) < \varepsilon$ implies $|(b^\rho/\rho)\, \mu(E^c)^{1-\rho}| < \varepsilon^{1-\rho} \, |b^\rho/\rho|$, and so for sufficiently small ε,

$$\int_{E^c} \frac{1}{\rho} (x^t(\tau))^\rho \, d\tau < \alpha/8 \tag{7}$$

since $1 - \rho > 0$.

Finally, free disposal implies

$$\int_{E^c} \frac{1}{\rho} (z^t(\tau))^\rho \, d\tau \geqslant \int_{E^c} \frac{1}{\rho} (\underline{z})^\rho \, d\tau = \frac{1}{\rho} (\underline{z})^\rho \, \mu(E^c).$$

But $|(1/\rho)(\underline{z})^\rho \, \mu(E^c)| < \varepsilon \, |(1/\rho)(\underline{z})^\rho|$, and so for sufficiently small ε,

$$\int_{E^c} \frac{1}{\rho} (z^t(\tau))^\rho \, d\tau > -\alpha/8.$$

Hence, utility inequalities (5) and (7) imply

$$\int_{B_t \cap E} \frac{1}{\rho} [(x^t(\tau) + z^t(\tau))^\rho - (x^t(\tau))^\rho] \, d\tau$$
$$+ \int_{E^c} \frac{1}{\rho} [(z^t(\tau))^\rho - (x^t(\tau))^\rho] \, d\tau > \alpha/4 - \alpha/8 - \alpha/8 = 0,$$

which is the desired result (4). ∎

5. EXISTENCE STATEMENT

EXISTENCE THEOREM. *An equilibrium exists under standard assumptions A.1 through A.6, plus storage A.7, when each preference order is surge proper and technology is Markovian.*

The existence theorem for commodity flows in L_1 can be strengthened to bound flows, restricting consumption and production to the smaller space $L_\infty \subset L_1$, if we suitably strengthen surge properness to further limit the contribution of intense but brief consumption flows. The theorem also extends to non-storable goods, like continuous labor input, as long as one bounds the marginal rate of substitution between labor and some storable good.

6. EXTRACTING A LIMIT OF DISCRETE APPROXIMATIONS

This section begins the proof of the Existence Theorem like the Arrow–Debreu literature, approximating each continuous-time economy with a net of discrete-time subeconomies. This section extracts a candidate for continuous-time equilibrium consumption from a limit of subeconomy equilibria so that utility from the candidate equals the limit of utility from subequilibria. Throughout, we exploit the nature of discrete goods by restricting, without loss of generality, endowments and all consumption and production of each discrete good to be constant within each time period.

Approximate any economy $\mathscr{E}$ satisfying the hypothesis of the Existence Theorem with a net $\{\mathscr{E}^\alpha\}$ of subeconomies indexed by pairs

$$\alpha = (\mathbb{E}, F),$$

where $\mathbb{E}$ is a finite collection of measurable subsets E of $[0, 1)$, and where F is a finite-dimensional subset of individual commodity streams in $L_1[0, 1)$ such that F contains each endowment flow e_i^{nt} of each consumer for each good in each period. Direct the net of subeconomies by inclusion, ordering indices $\alpha \geqslant \alpha'$ if $\mathbb{E} \supset \mathbb{E}'$ and $F \supset F'$. Define the commodity space L^α for economy $\mathscr{E}^\alpha$ as the subset of L_1^{N2} spanned by vectors of the truncated form z_E, as defined in Section 4, for some E in $\mathbb{E}$ and some vector z such that $z^{nt} \in F$ for each good and each period. Hence, define subeconomy $\mathscr{E}^\alpha$ by keeping the same endowments as $\mathscr{E}$, but reducing consumption sets from X_i to $X_i^\alpha := X_i \cap L^\alpha$, and reducing production sets from Y_g to $Y_g^\alpha := Y_g \cap L^\alpha$.

Since the original economy $\mathscr{E}$ satisfies the standard assumptions and the commodity space for each generation in each subeconomy is finite dimensional, each subeconomy $\mathscr{E}^\alpha$ is isomorphic to an economy satisfying the counterparts to standard assumptions in discrete time. Hence, discrete-time existence theorems, originally stated for pure-exchange but readily generalized to production [5, 8, 18], imply $\mathscr{E}^\alpha$ has an equilibrium. (That is, the natural analogs of all equilibrium conditions are satisfied when consumption sets and production sets are reduced as in $\mathscr{E}^{\alpha}$.) According to the Hahn–Banach extension theorem, the equilibrium price sequence for $\mathscr{E}^{\alpha}$ extends to an element of P, and the subeconomy equilibrium can be written

$$(x^{\alpha}, y^{\alpha}, p^{\alpha}) \in X \times Y \times P.$$

Storage (A.7) and profit maximization implies consumer 1's wealth is positive. Hence, normalize the price sequence p^{α} so that consumer 1 has unit wealth, $(p^{1\alpha}, p^{2\alpha}) \cdot e_1 = 1$.

LEMMA 1. *There exists some subnet* $\{(x^{\alpha}, y^{\alpha}, p^{\alpha})\}$ *of subequilibrium that converges to a limit* (x, y, p) *in* $X \times Y \times P$ *in the following sense*:

C.1. *For each period,* $p^{t\alpha}$ *(weak)* $\sigma\langle L_{\infty}^{N}, L_1^{N}\rangle$*-converges to* p^t. $(p^g, p^{g+1})\ Y_g \leqslant 0$ *for each generation.*

C.2. *For each consumer and for each positive tolerance* ε, *there exists a subset* E *of* $[0, 1)$ *with measure* $\mu(E) > 1 - \varepsilon$ *such that* x_{iE}^{α} *(weak)* $\sigma\langle L_1^{N2}, L_{\infty}^{N2}\rangle$*-converges to* x_{iE}. $u_i(x_i) \geqslant \lim\ \inf u_i(x_i^{\alpha})$.

C.3. *For each generation, each discrete good, and each period, the constant flow* $y_g^{nt\alpha}$ *converges to the constant flow* y_g^{nt}.

And the limit (x, y) *is an allocation.*

The proof of Lemma 1 shows how using truncated consumption (C.2) removes spikes from limit consumption flows.

Proof of C.1. According to the Tychonoff Theorem [12, page 232], for convergence it suffices to show, for each vector s in L_1^N, the value net $\{p^{nt\alpha} s^n\}$, for each good, is bounded from above. (If all value nets are bounded above, then all are also bounded below since the negative of each net is another net.) The storage set S_t defined in assumption A.7 for good n and time t has a non-empty interior (Observation 1) and therefore, without loss of generality, $s^n \in S_t$.

For the first period $t = 1$, storage (A.7) implies

$$e_1^1 + \bar{y}_1^1 = (0, ..., s^n, 0, ...)$$

for some production $\bar{y}_1$ in Y_1. Hence, the non-negativity of price and endowment, the truncation $(\bar{y}_1^1, 0) \in Y_1$ (A.4), and the non-positivity of profit from production cones (A.4) imply, for α large enough so that $(\bar{y}_1^1, 0)$ is in the commodity space of subeconomy $\mathscr{E}^\alpha$,

$$p^{n1\alpha}s^n = p^{n1\alpha}(e_1^1 + \bar{y}_1^1) \leqslant (p^{1\alpha}, p^{2\alpha}) \cdot e_1$$

which with price normalization bounds value $p^{n1\alpha}s^n$ from above. The second-period value net $\{p^{n2\alpha}s^n\}$ is likewise bounded.

For each period $t > 2$, storage (A.7) implies

$$e_1^1 + \bar{y}_1^1 = 0,\ e_1^2 + \bar{y}_1^2 + \bar{y}_2^2 = 0,\ \bar{y}_2^3 + \bar{y}_3^3 = 0, ..., \bar{y}_{t-1}^t + \bar{y}_t^t = (0, ..., s^n, 0, ...)$$

for some list $\bar{y}_1$ in Y_1 through $\bar{y}_t$ in Y_t. Hence, adding together the resulting sequence

$$\begin{aligned} &p^{1\alpha}e_1^1 + p^{1\alpha}\bar{y}_1^1 = 0, \qquad && p^{2\alpha}e_1^2 + p^{2\alpha}\bar{y}_1^2 + p^{2\alpha}\bar{y}_2^2 = 0, \\ &p^{3\alpha}\bar{y}_2^3 + p^{3\alpha}\bar{y}_3^3 = 0, ..., && p^{t\alpha}\bar{y}_{t-1}^t + p^{t\alpha}\bar{y}_t^t = p^{nt\alpha}s^n \end{aligned}$$

of value equalities yields

$$\begin{aligned} p^{nt\alpha}s^n = {} & (p^{1\alpha}, p^{2\alpha}) \cdot e_1 + (p^{1\alpha}, p^{2\alpha}) \cdot \bar{y}_1 \\ & + (p^{2\alpha}, p^{3\alpha}) \cdot \bar{y}_2 + \cdots + (p^{t\alpha}, p^{t+1,\alpha}) \cdot (\bar{y}_t^t, 0). \end{aligned}$$

Hence, truncation $(\bar{y}_t^t, 0) \in Y_t$ (A.4) and the non-positivity of profit from production cones (A.4) imply, for α large enough so that the entire production list is in the commodity space of subeconomy $\mathscr{E}^\alpha$,

$$p^{nt\alpha}s^n \leqslant (p^{1\alpha}, p^{2\alpha}) \cdot e_1$$

which with price normalization bounds value $p^{nt\alpha}s^n$ from above.

Given price convergence, $(p^g, p^{g+1})\ Y_g \leqslant 0$ follows at the limit since subequilibrium profit maximization implies $(p^{g\alpha}, p^{g+1,\alpha})\ Y_g^\alpha \leqslant 0$. ▮

Proof of C.2 *and* C.3. For $L_\infty^{N2\prime}$ equal to the norm dual of L_∞^{N2}, according to the Alaoglu theorem, L_1^{N2}-bounded subsets of $L_\infty^{N2\prime}$ are $\sigma\langle L_\infty^{N2\prime}, L_\infty^{N2}\rangle$ – compact.

Hence, the boundedness of feasible consumption (A.5) implies, for each consumer, some subnet of subequilibrium consumption $\sigma\langle L_\infty^{N2\prime}, L_\infty^{N2}\rangle$-converges to an element of $L_\infty^{N2\prime}$. Since that limit is non-negative, the Yosida–Hewitt theorems [19, Theorem 1.19, Theorem 1.23] imply the limit is of the form $x_i + q_i$, where $x_i \in X_i$ but $q_i \in L_\infty^{N2\prime}$ is purely finitely additive in the sense that, for each positive tolerance ε, there exists a subset E of $[0, 1)$ with measure $\mu(E) > 1 - \varepsilon$ such that $q_i^t(\tau) = 0$ for each period t and each τ in E. Hence, x_{iE}^α (weak) $\sigma\langle L_1^{N2}, L_\infty^{N2}\rangle$-converges to x_{iE}.

Likewise, there exists some further subnet so that production converges in the manner of consumption. Namely, there is a y in Y specifying production that is a limit in the sense:

C.3′. *For each generation and for each positive tolerance* ε, *there exists a subset* E *of* $[0,1)$ *with measure* $\mu(E)>1-\varepsilon$ *such that* y^{α}_{gE} *(weak)* $\sigma\langle L_1^{N2}, L_\infty^{N2}\rangle$*-converges to* y_{gE} *for truncated production*

$$y^{nt}_{gE}(\tau):=\begin{cases}0 & \textit{if good } n \textit{ is continuous and } \tau\notin E\\ y^{nt}_g(\tau) & \textit{otherwise.}\end{cases}$$

In particular, the convergence of truncated production vectors (C.3′) implies the convergence of production of each discrete good (C.3).

It only remains to prove $u_i(x_i)\geqslant \liminf u_i(x_i^\alpha)$. To that end, for each vector $z\gg 0$ in L_1^{N2}, consumer i's surge properness implies there exists a positive ε such that

$$x^{\alpha}_{iE}+z\succ_i x^{\alpha}_i \tag{8}$$

for each subequilibrium consumption x_i^α and for the set $E\subset[0,1)$ in C.2 above with measure $\mu(E)>1-\varepsilon$. But the convergence of consumption implies weak convergence $(x^\alpha_{iE}+z)\to(x_{iE}+z)$, and since preferences are convex (A.3), Mackey upper semi-continuity (A.2) implies weak upper semi-continuity, according to the Hahn–Banach theorem. Hence,

$$u_i(x_{iE}+z)\geqslant \liminf u_i(x^\alpha_{iE}+z).$$

Hence, $x_i\geqslant x_{iE}$, free disposal (A.1), and surge properness (8) imply

$$u_i(x_i+z)\geqslant u_i(x_{iE}+z)\geqslant \liminf u_i(x_i^\alpha)$$

which holding for each $z\gg 0$ in X_i implies

$$u_i(x_i)\geqslant \liminf u_i(x_i^\alpha)$$

since preferences are Mackey upper semi-continuous (A.2). ∎

Finally, the limit (x, y) balances materials (1), and so qualifies as an allocation, because each subequilibrium (x^α, y^α) is an allocation, and truncated consumption (C.2) and truncated production (C.3′) converge.

7. BUDGET CONSTRAINTS AND PROFIT MAXIMIZATION

This section finishes the existence proof, showing that candidate consumption and production satisfy equilibrium budget constraints and profit maximization.

LEMMA 2. *For the subnet of subequilibria described in Lemma* 1, *the expenditure of each consumer and profit of each generation converge*

$$p^{t\alpha} x_i^{t\alpha} \to p^t x_i^t; \qquad p^{t\alpha} y_g^{t\alpha} \to p^t y_g^t$$

in each period t of life.

Proof of Theorem 2 (using Lemma 2). The desired equilibrium is the subequilibrium limit described in Lemma 1. In particular, Lemma 2 implies that the limit preserves budget constraints and has production that generates zero profits. Hence, the absense of positive profit (C.1) implies profit maximization. It only remains to prove utility maximization.

To that end, first prove maximization for each consumer with positive wealth, $(p^g, p^{g+1}) \cdot e_i > 0$. Consider any consumption $\hat{x}$ generating greater utility than the limit consumption, $u_i(\hat{x}) > u_i(x_i)$. Euclidean lower semi-continuity (A.2) implies $u_i(\lambda \hat{x}) > u_i(x_i)$ for some $\lambda < 1$. Hence, utility semi-convergence (C.2) implies $u_i(\lambda \hat{x}) > u_i(x_i^\alpha)$ for large α. Hence, sub-equilibrium utility maximization implies the higher utility is unaffordable, $(p^{g\alpha}, p^{g+1,\alpha}) \cdot (\lambda \hat{x}) > (p^{g\alpha}, p^{g+1,\alpha}) \cdot e_i$. Hence, price convergence (C.1) implies $(p^g, p^{g+1}) \cdot (\lambda \hat{x}) \geqslant (p^g, p^{g+1}) \cdot e_i$, and positive wealth with $\lambda < 1$ implies $(p^g, p^{g+1}) \cdot \hat{x} > (p^g, p^{g+1}) \cdot e_i$. In summary, any consumption $\hat{x}$ generating greater utility than the limit consumption is unaffordable, which with budget constraints constitutes utility maximization.

It only remains to prove each consumer has positive wealth. Consider proof by contradiction; assume the set of consumers with zero wealth is non-empty. Subequilibrium price normalization and the price convergence (C.1) implies the set of consumers with positive wealth is also non-empty. Hence, applying irreducibility to the limit allocation (x, y) yields at least one consumer i with positive wealth that desires the endowments of the consumers with zero wealth (A.6): that is, $x_i + z \succ_i x_i$ for some $z \leqslant \sum_{\hat{\imath}} e_{\hat{\imath}}$. But consumer i's utility maximization implies z has positive value, $(p^g, p^{g+1}) \cdot z > 0$. Hence, $z \leqslant \sum_{\hat{\imath}} e_{\hat{\imath}}$ implies positive wealth for at least one of the finite number of (alleged) zero-wealth consumers $\hat{\imath}$ whose generation overlaps with that of consumer i. Contradiction. ∎

Proof of Lemma 2. Since price converges (C.1), it suffices to prove expenditure and profit convergence

$$p^{nt\alpha}(x_i^{nt\alpha} - x_i^{nt}) \to 0; \qquad p^{nt\alpha}(y_g^{nt\alpha} - y_g^{nt}) \to 0 \tag{9; 10}$$

for each good. To that end, for each discrete good, expenditure convergence follows from the convergence of price (C.1) and consumption (C.2), and profit convergence, from price (C.1) and production (C.3). Hence, show convergence for each continuous good n.

Since technology is Markovian, material balance (1) at each subequilibrium and at the limit implies

$$y_g^{nt\alpha} - y_g^{nt} = \sum_i (x_i^{nt\alpha} - x_i^{nt}) \quad \text{if } t = g; \qquad y_g^{nt\alpha} - y_g^{nt} = 0 \quad \text{if } t = g+1$$

for each index α. Hence, profit convergence (10) follows from expenditure convergence (9) for each consumer, meaning it suffices to prove expenditure convergence (9). To simplify that proof, fix the consumer in generation g and drop the consumer subscript throughout.

For any measurable subset $E \subset [0, 1)$, the triangle inequality

$$|p^{nt\alpha}(x^{nt\alpha} - x^{nt})| \leqslant |p^{nt\alpha}(x^{nt\alpha} - x_E^{nt\alpha})| + |p^{nt\alpha}(x_E^{nt\alpha} - x_E^{nt})| + |p^{nt\alpha}(x_E^{nt} - x^{nt})| \tag{11}$$

implies the convergence of expenditure (9) if we prove the convergence of each term on the right. Precisely, for each positive ε, we find a suitable set E so that, for large α, each term on the right is less than ε.

To bound the first term in the triangle inequality (11), fix any $z \geqslant 0$ in X_i costing $(p^g, p^{g+1}) \cdot z < \varepsilon$. Surge properness implies there exists a positive tolerance δ such that, for each subset $E \subset [0, 1)$ with measure $\mu(E) > 1 - \delta$, preference $x_E^\alpha + z \succ x^\alpha$ for all α. But for large α, x_E^α and z are both in the commodity space of $\mathscr{E}^\alpha$; hence, utility maximization by subequilibrium consumption x^α implies

$$(p^{g\alpha}, p^{g+1,\alpha}) \cdot (x_E^\alpha + z) > (p^{g\alpha}, p^{g+1,\alpha}) \cdot x^\alpha$$

which with $x^\alpha \geqslant x_E^\alpha$ and the non-negativity of price (A.1) implies

$$|p^{nt\alpha}(x^{nt\alpha} - x_E^{nt\alpha})| \leqslant (p^{g\alpha}, p^{g+1,\alpha}) \cdot (x^\alpha - x_E^\alpha) < (p^{g\alpha}, p^{g+1,\alpha}) \cdot z.$$

But from $(p^g, p^{g+1}) \cdot z < \varepsilon$, the convergence of price (C.1) implies $(p^{g\alpha}, p^{g+1,\alpha}) \cdot z < \varepsilon$ for large α. In summary, for each set satisfying $\mu(E) > 1 - \delta$, we have $|p^{nt\alpha}(x^{nt\alpha} - x_E^{nt\alpha})| < \varepsilon$ for large α.

To bound the third term in the triangle inequality (11), compute the integral product

$$p^{nt}(x^{nt} - x_E^{nt}) = \int_{E^c} p^{nt}(\tau)\, x^{nt}(\tau)\, d\tau$$

over the complement E^c of any set E. Hence, there exists a positive tolerance (without loss of generality, the tolerance δ above) such that, for each set satisfying $\mu(E) > 1 - \delta$, measure $\mu(E^c) < \delta$ and the integral is less than ε [16, Proposition 4.14, p. 88]. Hence, the convergence of price (C.1) implies, for large α, subequilibrium prices preserve the integral inequality, $p^{nt\alpha}(x_E^{nt} - x^{nt}) < \varepsilon$.

Finally, to bound the middle term in the triangle inequality (11), fix constant $b > 1/\varepsilon$. For the δ fixed from bounding the first and third inequality terms, define the subset E of $[0, 1)$ with measure $\mu(E) > 1-\delta$ described in the convergence of consumption (C.2). Weak convergence $x^{\alpha}_{iE} \to x_{iE}$ implies weak convergence $bx^{nt\alpha}_{E} \to bx^{nt}_{E}$. To proceed, use the following:

Lemma 3. *Consider any (weak) $\sigma\langle L_1, L_\infty\rangle$-convergent net $z^\alpha \to z$ of non-negative flows and any storage set $S = S(s_0, \bar{s}, \delta)$.*
Then, for large α, the set $S = S(s_0, \bar{s}, \delta)$ contains both $(z^\alpha - z)$ and $(z - z^\alpha)$.

Lemma 3 implies, for large α, the storage set S from assumption A.7, for good n and period t, contains both $s^{n\alpha} := bx^{nt\alpha}_{E} - bx^{nt}_{E}$ and its negative. Hence deduce $|p^{nt\alpha}s^{n\alpha}| \leqslant 1$ for large α.

If $t = 1$, storage (A.7) implies, for $s^{n\alpha}$ in S,

$$e^1_1 + \bar{y}^1_1 = (0, ..., s^{n\alpha}, 0, ...) \tag{12}$$

for some production $\bar{y}_1$ in Y_1. Hence show, for large α, the truncation $(\bar{y}^1_1, 0)$ is contained in the subeconomy production set, Y^α_1. To that end, recall from Section 6 the definition $\alpha = (\mathbb{E}, F)$ of indices and the commodity space of each subeconomy, $\mathscr{E}^\alpha$. For the continuous good n, the storage equality (12) implies $\bar{y}^{n1}_1 = -e^{n1}_1 + s^{n\alpha}$, which is in F because $s^{n\alpha} \in F$ for large α. For each other continuous good $m \neq n$, the storage equality (12) implies $\bar{y}^{m1}_1 = -e^{m1}_1$, which is also in F. Hence, for large α, $(\bar{y}^1_1, 0) \in Y^\alpha_1$ and, therefore, the non-positivity of subequilibrium profit from subeconomy production cones (A.4) implies $p^{1\alpha}\bar{y}^1_1 \leqslant 0$. Hence, the storage equality (12) implies

$$p^{n1\alpha}s^{n\alpha} \leqslant p^{1\alpha}e^1_1.$$

Hence, price normalization with the non-negativity of price (A.1) and endowment implies $p^{n1\alpha}s^{n\alpha} \leqslant 1$. But repeating the argument for the negative flow $-s^{n\alpha}$ implies $-p^{n1\alpha}s^{n\alpha} \leqslant 1$ and so $|p^{n1\alpha}s^{n\alpha}| \leqslant 1$, for large α. The second-period inequality $|p^{n2\alpha}s^{n\alpha}| \leqslant 1$ likewise follows for large α.

If $t > 2$, storage (A.7) implies, for $s^{n\alpha}$ in S,

$$e^1_1 + \bar{y}^1_1 = 0, \qquad e^2_1 + \bar{y}^2_1 + \bar{y}^2_2 = 0,$$
$$\bar{y}^3_2 + \bar{y}^3_3 = 0, ..., \qquad \bar{y}^t_{t-1} + \bar{y}^t_t = (0, ..., s^{n\alpha}, 0, ...)$$

for some list $\bar{y}_1$ in Y_1 through $\bar{y}_t$ in Y_t. Hence, adding together the resulting sequence

$$p^{1\alpha}e^1_1 + p^{1\alpha}\bar{y}^1_1 = 0, \qquad p^{2\alpha}e^2_1 + p^{2\alpha}\bar{y}^2_1 + p^{2\alpha}\bar{y}^2_2 = 0,$$
$$p^{3\alpha}\bar{y}^3_2 + p^{3\alpha}\bar{y}^3_3 = 0, ..., \qquad p^{t\alpha}\bar{y}^t_{t-1} + p^{t\alpha}\bar{y}^t_t = p^{nt\alpha}s^{n\alpha}$$

of value equalities yields

$$p^{nt\alpha}s^{n\alpha} = (p^{1\alpha}, p^{2\alpha}) \cdot e_1 + (p^{1\alpha}, p^{2\alpha}) \cdot \bar{y}_1 + (p^{2\alpha}, p^{3\alpha}) \cdot \bar{y}_2 + \cdots + (p^{t\alpha}, p^{t+1,\alpha}) \cdot (\bar{y}^t_t, 0).$$

But the containment argument above shows, for large α, each member of the production list $\bar{y}_1$ through $\bar{y}_t$ and the truncation $(\bar{y}^t_t, 0)$ are contained in the subeconomy production sets for the corresponding generations, $\bar{y}_1 \in Y^\alpha_1$ through $\bar{y}_t \in Y^\alpha_1$ and $(\bar{y}^t_t, 0) \in Y^\alpha_t$. Hence each profit is non-negative and, therefore,

$$p^{nt\alpha}s^{n\alpha} \leqslant (p^{1\alpha}, p^{2\alpha}) \cdot e_1.$$

Hence, price normalization implies $p^{nt\alpha}s^{n\alpha} \leqslant 1$. But repeating the argument for the negative flow $-s^{n\alpha}$ implies $-p^{nt\alpha}s^{n\alpha} \leqslant 1$ and so $|p^{nt\alpha}s^{n\alpha}| \leqslant 1$, for large α.

Given $|p^{nt\alpha}s^{n\alpha}| \leqslant 1$, the definition of $s^{n\alpha}$ implies $b\,|p^{nt\alpha}(x^{nt\alpha}_E - x^{nt}_E| \leqslant 1$, and the definition $b > 1/\varepsilon$ implies $|p^{nt\alpha} \cdot (x^{nt\alpha}_E - x^{nt}_E)| < \varepsilon$. That bounds the middle term in the triangle inequality (11).

Putting it all together, it only remains to prove Lemma 3.

Proof of Lemma 3. Showing $(z^\alpha - z) \in S$ or $(z - z^\alpha) \in S$ means showing the flows satisfy the storage inequalities (2). To simplify, reduce (if necessary) the initial stock s_0 in the storage set so that $s_0 < \bar{s}$. Hence, flows $(z^\alpha - z)$ and $(z - z^\alpha)$ satisfy the storage inequalities whenever the storage bound

$$\left| \int_0^t e^{\delta\tau}(z^\alpha - z)(\tau)\, d\tau \right| \leqslant 3r \tag{13}$$

is satisfied for t in $[0, 1)$, where $r := 1/3 \min\{s_0, \bar{s} - s_0\}$.

To show the storage bound, since $z \in L_1$, there exists a subdivision $\Delta = \{c_0, c_1, ..., c_N\}$ of the unit interval $[0, 1]$ such that [16, Proposition 4.14, p. 88]

$$\int_{c_n}^{c_{n+1}} e^{\delta\tau} z(\tau)\, d\tau < r \tag{14}$$

for each subinterval $[c_n, c_{n+1}]$. Hence, the $\sigma\langle L_1, L_\infty\rangle$-convergence of z^α to z implies, for large α,

$$\int_{c_n}^{c_{n+1}} e^{\delta\tau} z^\alpha(\tau)\, d\tau < r \tag{15}$$

for each subinterval $[c_n, c_{n+1}]$. Meanwhile, the $\sigma\langle L_1, L_\infty\rangle$-convergence of z^α to z implies, for large α,

$$\left|\int_0^{c_n} e^{\alpha\tau}(z^\alpha - z)(\tau)\, d\tau\right| < r \tag{16}$$

for each interval $[0, c_n]$.

Putting it all together, consider any α large enough to satisfy inequalities (15) and (16). Fix any t in $[0, 1)$. Select a subinterval $[c_n, c_{n+1}]$ containing t. Since

$$\int_0^t e^{\delta\tau}(z^\alpha - z)(\tau)\, d\tau = \int_0^{c_n} e^{\delta\tau}(z^\alpha - z)(\tau)\, d\tau + \int_{c_n}^t e^{\delta\tau} z^\alpha(\tau)\, d\tau - \int_{c_n}^t e^{\delta\tau} z(\tau)\, d\tau,$$

the triangle inequality yields the storage bound (13) from the inequality (16) and the inequalities

$$\left|\int_{c_n}^t e^{\delta\tau} z^\alpha(\tau)\, d\tau\right| < \varepsilon \qquad \text{and} \qquad \left|\int_{c_n}^t e^{\delta\tau} z(\tau)\, d\tau\right| < \varepsilon.$$

There the first inequality follows from inequality (15) since $z^\alpha \geqslant 0$, and the second from inequality (14) since $z \geqslant 0$. ∎

8. NON-EXISTENCE EXAMPLE

This section contains the non-existence example showing that the assumptions of Markovian technology and storage cannot be dropped. In fact, not only does the non-existence example satisfy all our assumptions except Markovian technology and storage, but preferences are linear and so satisfy the additional continuity assumptions in the overlapping-generations and Arrow–Debreu literature, and technology satisfies Zame's additional assumption of bounded marginal rates of transformation [20].

The economy has 1 good, and 1 consumer per generation. Generation t has linear utility

$$u_t(x_t) = \int_0^{1/2} x_t^t(\tau)\, d\tau + 2 \int_0^{1/2} x_t^{t+1}(\tau)\, d\tau$$

and endowment

$$e_t = (e_t^t, e_t^{t+1}) = (\theta^1, \theta^1),$$

where $\theta^1 := \chi_{[0,\,1/2)} \in L_1[0, 1)$ indicates 1 unit of consumption flow during the first half of the period. Before defining production technology, define unit flows

$$\theta^k := \chi_{[1-2^{1-k},\,1-2^{-k})} \in L_1[0, 1)$$

over each subinterval $[1-2^{1-k}, 1-2^{-k})$ in the partition

$$[0, 1/2), \qquad [1/2, 3/4), \ldots, \qquad [1-2^{1-k}, 1-2^{-k}), \ldots,$$

of the unit interval $[0, 1)$.

Generation t has production set

$$Y_t := \left\{(y_t^t, y_t^{t+1}) \leqslant \sum_{k=1}^{\infty} a^k(-2\theta^{k+1}, \theta^k)\text{: for some sequence } a^k \geqslant 0\right\}$$

with the production set of generation 1 the sum of Y_1 above and the truncated set

$$Y_0 := \left\{(y_1^1, 0) \leqslant \sum_{k=1}^{\infty} a^k(-\theta^k + 2\theta^{k+1}, 0)\text{: for some sequence } a^k \geqslant 0\right\}.$$

Y_0 embodies a particular storage technology within period 1, while each other set Y_t embodies a time-to-build technology, spanning periods t and $t+1$. For instance, $(-\theta^1 + 2\theta^2, 0) \in Y_0$ indicates transformation within period 1 and $(-2\theta^2, \theta^1) \in Y_1$ indicates transformation from period 1 to period 2.

OBSERVATION 5. *The economy $\mathscr{E}$ satisfies the standard assumptions plus preferences are surge proper. And each production set satisfies Zame's additional assumption of bounded marginal rates of transformation.*[5]

Yet, the economy $\mathscr{E}$ has no equilibrium.

Proof. Confirming the economy satisfies the indicated assumptions is unremarkable. To prove the economy has no equilibrium, assume it has, then get a contradiction.

Convert the equilibrium, without loss of generality, so that every consumption is a step function within the first half of each life period, and is zero elsewhere. That is, generation t consumes

$$x_t = (x_t^t, x_t^{t+1}) = (\bar{x}_t^t\theta^1, \bar{x}_t^{t+1}\theta^1)$$

[5] In fact, the marginal rate of technological transformation is bounded by 1: for any $y = y^+ - y^-$ in Y_t and any vector $0 \leqslant \hat{y}^- \leqslant y^-$, then there exists a $\hat{y}^+$ such that $0 \leqslant \hat{y}^+ \leqslant y^+$, $\hat{y} := \hat{y}^+ - \hat{y}^- \in Y_t$, and $\|y^+ - \hat{y}^+\| \leqslant \|y^- - \hat{y}^-\|$.

for some scalers $\bar{x}_t^t$ and $\bar{x}_t^{t+1}$, which generates utility

$$u_t(x_t) = \bar{x}_t^t/2 + \bar{x}_t^{t+1}.$$

Hence, market clearing implies time t total production $y^t = \sum_g y_g^t$ is of the form

$$y^t = \bar{y}^t \theta^1$$

and satisfies

$$\bar{x}_1^1 = 1 + \bar{y}^1$$

for consumption during the first half of period 1, and

$$\bar{x}_{t-1}^t + \bar{x}_t^t = 2 + \bar{y}^t$$

for each later period.

The definition of production implies $p^t\theta^1 \leqslant p^1\theta^1$, for each period. For instance, if $t=2$, the containments $(-\theta^1 + 2\theta^2, 0) \in Y_0$ and $(-2\theta^2, \theta^1) \in Y_1$ and the non-positivity of profit from production cones implies $p^1(-\theta^1 + 2\theta^2) \leqslant 0$ and $-2p^1\theta^2 + p^2\theta^1 \leqslant 0$ and, therefore, $-p^1\theta^1 + p^2\theta^1 \leqslant 0$.

In fact, using strong induction, we prove constant price $p^t\theta^1 = p^1\theta^1$, for each period t. Assume constancy for all periods before t. From the previous inequality $p^t\theta^1 \leqslant p^1\theta^1$, the only alternative to the desired equality is $p^t\theta^1 < p^1\theta^1$. But in that case, the induction hypothesis of price constancy for all periods before t implies $p^t\theta^1 < p^{t-1}\theta^1$. Hence, utility maximization implies generation $t-1$ sells all young-age consumption and buys more than 2 units of old-age consumption, $\bar{x}_{t-1}^t > 2$. Hence, market clearing implies total production $\bar{y}^t > 0$. Hence, equilibrium production can be rearranged to produce $\bar{y}^t\theta^1$ less in period t and $\bar{y}^t\theta^1$ more in period 1. But equilibrium profit maximization implies such rearrangements do not increase profit, meaning $p^t(\bar{y}^t\theta^1) \geqslant p^1(\bar{y}^t\theta^1)$, and $p^t\theta^1 \geqslant p^1\theta^1$, which contradicts the alternative $p^t\theta^1 < p^1\theta^1$.

Given price constancy, $p^t\theta^1 = p^1\theta^1$, utility maximization by generation t determines lifetime consumption, $(\bar{x}_t^t, \bar{x}_t^{t+1}) = (0, 2)$. Hence, market clearing determines total production, $(\bar{y}^1, \bar{y}^2, \bar{y}^3, ...) = (-1, 0, 0, ...)$. But, from the definition of the production sets, the only way to achieve that total is through the generational production sequence $y_0 = (-\theta^1, 0)$, $y_1 = (0, 0)$, ... But generation 1 inputs (valuable) goods and never outputs, which contradicts profit maximization. ▮

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