On the Existence of
Price Equilibria in Dynamic Economies*

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I extend McKenzie's general characterization of the existence of competitive equilibria in static (finite horizon) models to a class of dynamic (infinite horizon) economies that subsumes overlapping-generations as a special case. With a few necessary exceptions, the assumptions employed in my existence proof are natural extensions of McKenzie's. In particular, I strengthen the existing literature by imposing McKenzie-type irreducibility only on the whole economy and not on certain (artificial) sub-economies and by imposing McKenzie's weak continuity and convexity conditions. I also establish the general existence of equilibria with nominal government surplus. Journal of Economic Literature Classification Numbers: 021, 023, 111. © 1988 Academic Press, Inc.

1. INTRODUCTION

Perhaps the fundamental problem in the study of competitive markets is to characterize the conditions under which price equilibria exist. For years, classification attempts were restricted to static models; that is, models involving only finite numbers of agents and goods and, hence, presupposing the existence of some (arbitrary) terminal date.

Recently, however, many find that in order to explain some basic market phenomena, such as the use of fiat (unbacked) money in an economy with fully informed and rational agents, one must pass to genuinely dynamic models; precisely, models such as overlapping-generations that involve a double infinity of agents and (dated) goods. Only recently has a characterization of those economies with price equilibria been attempted in a dynamic setting. I strengthen this foundation both by weakening the assumptions known to guarantee the existence of equilibria and by uncovering, via counter-examples, previously unknown conditions that are essential to the existence of price equilibria.

The early competitive equilibrium existence theorems of Arrow and

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Debreu [1], McKenzie [11], and Nikaido [14]—for static models—were refined over the years as the assumptions needed to guarantee existence were successively weakened. McKenzie [12] provides a review of this line of research as well as, what appears to be, the minimal (weakest) set of sufficient conditions known to date. For dynamic models, Balasko et al. [2] provide an existence theorem for overlapping-generations economies that is generalized by Wilson [15] to allow both weaker regularity assumptions and a more general demographic structure.

One of the assumptions employed in the proofs of Balasko et al. and Wilson that has no natural counterpart in McKenzie is their irreducibility condition. Not only do they impose McKenzie's irreducibility condition on the entire economy, but they require the existence of a certain collection of finite sub-economies each of which is irreducible. In this paper, I refine their proof techniques to eliminate the need for this unnatural sub-economy irreducibility assumption.¹ In addition, I weaken their continuity and convexity restrictions on preferences to the point that they are analogous to McKenzie's.

I follow Wilson [15] and require the consumption spaces of infinitely lived agents to consist of all non-negative bundles (vectors). This assumption is stronger than in McKenzie's static model, where consumption spaces are arbitrary closed and convex collections of non-negative bundles. However, I provide a counter-example demonstrating that existence is no longer guaranteed when one considers arbitrary closed and convex sets for infinitely lived agents—even in the presence of an additional free disposal condition.

Finally, I adopt the requirement—implicit in Balasko et al. [2] and essential to Wilson [15]—that there be only a finite number of agents desiring (having tastes for) any particular good.² Again, I provide a counter-example demonstrating the necessity of such a restriction.

I turn my attention next to establishing equilibria with income transfers. This classic problem, as posed by Hahn [9] and later examined by Cass and Yaari [3] among (many) others, concerns equilibria in which agents receive positive transfers (fiat money). However, as is now well known and as my Proposition 5 reiterates, the existence of equilibria with positive transfers is not robust even to variations in endowment streams; in particular, a Pareto efficient autarkic distribution rules out positive transfer equilibria. In contrast, my Proposition 6 establishes the general abundance of equilibria with negative income transfers (taxes).

¹ Burke [3] provides an example of an economy satisfying all my restrictions but not sub-economy irreducibility.

² This restriction was not brought out in Wilson [15].
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In this paper, I characterize the existence of both competitive and taxation equilibria for a class of dynamic economies that subsume overlapping-generations as a special case. In Section 2, I introduce my general model with its definitions and assumptions. In Section 3, I establish the compactness of equilibrium prices and allocations. In Section 4, I use compactness to establish the general existence of competitive equilibria. In Section 5, I demonstrate by way of counter-examples that my restrictions—when stronger than those employed in static models—are necessarily so. In Section 6, I return to my compactness result to establish the general existence of equilibria with negative income transfers.

For reference, I extend the techniques developed here to dynamic production economies in Burke [4].

2. THE MODEL

I begin with the standard formulation of a pure exchange economy modified to accommodate an infinite number of agents and goods as in Wilson [15]. Let \( A \) be a countable set of agents and, for each agent \( a \in A \), let \( X^a \) be his consumption set, \( (\succ) \) his preference relation over bundles in \( X^a \), and \( \omega^a \) his endowment. The set of commodities is countably infinite being indexed by the natural numbers. In the context of overlapping-generations, there are a finite number of (dated) commodities available in each of an infinite number of time periods.

I say that agent \( a \) does not desire good \( y \) when for any bundles \((x_1, \ldots, x_\gamma, \ldots), (y_1, \ldots, y_\gamma, \ldots) \in X^a\); \((y_1, \ldots, y_\gamma, \ldots) \succ (x_1, \ldots, 0, \ldots)\) implies \((y_1, \ldots, y_\gamma, \ldots) \succ (x_1, \ldots, x_\gamma, \ldots)\). An allocation scheme is an array \((x^a)\) such that \( x^a \in X^a \) for \( a \in A \). For convenience, I only allow agents to consume goods they desire. The collection of such schemes is thus

\[
X = \left\{ (x^a) \in \prod_{a \in A} X^a : \text{for } a \in A \text{ and } \gamma \geq 1, \right. \\
\left. x_\gamma^a = 0 \text{ if agent } a \text{ does not desire good } \gamma \right\}. \tag{1}
\]

Assumptions (2)--(10), below, are employed throughout this paper. With the exceptions of (2) and (7), they are analogous (or can be weakened to be analogous) to those of McKenzie [12]. Each of my assumptions are at least as weak as those of Wilson [15]; which are, in turn, at least as weak as those of Balasko et al. [2].

3 Clearly, given free disposal, restricting agents to consume only goods they desire may be satisfied in any equilibrium by resuffling allocations of free goods.
ASSUMPTION (regularity of consumption sets). For each \( a \in A \):\(^4\)

\[
X^a = \mathbb{R}^n_+ \tag{2}
\]

\[
\omega^a \in X^a. \tag{3}
\]

It may be somewhat surprising but, as I demonstrate by Example 1, (2) cannot be replaced by \( X^a \) being a closed and convex subset of \( \mathbb{R}^n_+ \) satisfying free disposal; that is, \( x \in X^a \) and \( y \geq x \) imply \( y \in X^a \).

I have developed an equilibrium existence proof that does not employ the survival assumption (3) but will not present it here due to its complexity.

For \( a \in A \) and \( x \in X^a \), let \( P^a(x) = \{ y \in X^a : y (\geq) x \} \) and \( P_{\geq}^a(x) = \{ y \in X^a : x (\geq) y \} \). \( P^a(x) \) are the bundles that agent \( a \) prefers to \( x \) while \( P_{\geq}^a(x) \) are those to which \( x \) is preferred.

**N.B. All subsets of \( \mathbb{R}^n_+ \) are endowed with the product topology.**\(^5\)

ASSUMPTION (regularity of preferences). For each \( a \in A \) and \( x \in X^a \):

\( P^a(x) \) and \( P_{\geq}^a(x) \) are both open relative to \( X^a \) \( \tag{4} \)

\( x \notin \text{convex hull of } P^a(x). \) \( \tag{5} \)

Assumption (4) is a slightly stronger continuity requirement than that imposed by McKenzie [12] for static economies but is weaker than the continuity condition employed in Wilson's existence proof for dynamic economies.\(^6\) For infinitely lived agents, (4) requires intertemporal discounting. For example, if preferences are represented by the weighted sum of instantaneous utilities \( \sum_{t \geq 1} (1 + \delta)^{-t} u(x_t) \), then (4) requires that the discount rate \( \delta \) be positive.\(^7\) Assumption (5) is the (clearly minimal) convexity condition imposed by McKenzie [12] and is weaker than Wilson's convexity conditions.\(^8\)

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\(^4\) I adopt the vector notation \( x \geq y \) if \( x_\gamma \geq y_\gamma \) for \( \gamma \geq 1 \) and \( x \succ y \) if \( x_\gamma > y_\gamma \) for \( \gamma > 1 \). Given this notation, I define \( R^a_+ = \{ x \in \mathbb{R}^n : x \geq 0 \} \) and \( R^a_\geq = \{ x \in \mathbb{R}^n : x \geq 0 \} \).

\(^5\) See Munkres [13, p. 113] for a definition of the product topology. In particular, convergence \( x(k) \to x \) in \( \mathbb{R}^n \) means pointwise convergence \( x_\gamma(k) \to x_\gamma \) for \( \gamma \geq 1 \).

\(^6\) Specifically, McKenzie weakens \( P_{\geq}^a(x) \) being relatively open at all \( x \in X^a \) to the correspondence \( P^a(\cdot) \) being lower semi-continuous. My techniques do allow me to employ a restricted (but unnatural) form of lower semi-continuity in place of the openness of \( P_{\geq}^a(x) \).

\(^7\) Wilson employs the openness of the graph of \( P(x) \) in Wilson [15, p. 103, line 17].

\(^8\) For \(-1 < \delta < 0\), one may employ the Weiszäcker overtaking criterion to induce a preference relation.

\(^9\) Wilson employs the additional convexity condition: if \( y \in P^a(x) \), then \( \lambda y + (1 - \lambda) x \in P^a(x) \) for \( 0 < \lambda < 1 \).
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ASSUMPTION (free disposal in preferences). For all \( a \in A \) and \( x, y \in X^a \);

\[ x \geq y \text{ implies } P^a(x) \subseteq P^a(y) \text{ and } P_{>1}^a(y) \subseteq P_{>1}^a(x). \]  

(6)

ASSUMPTION (finite number of consumers per good).

Only a finite number of agents desire any particular good. 

(7)

If commodities are dated, then (7) is satisfied when the population of agents is finite at every point in time. In Example 2, I demonstrate that it is essential to impose a condition like (7) to insure the existence of competitive equilibria. Technically, while (2) guarantees that individual demand is continuous, (7) insures that continuity is passed over to aggregate demand.\(^9\)

I assume that the total endowment of each good is finite and define \( \omega = \sum_{a \in A} x^a \) to be aggregate endowment of the economy. An allocation scheme \((x^a) \in X\) is called feasible if \( \sum_{a \in A} x^a = \omega \).

ASSUMPTION (positive aggregate endowments).

\[ \omega \geq 0. \]  

(8)

While the (free disposal) assumption (6) allows me to restrict attention to non-negative prices, (8) insures that prices are finite in an equilibrium.

ASSUMPTION (irreducibility of the entire economy). For any partition of \( A \) into two non-empty subsets \( A^0 \) and \( A^1 \) and any feasible allocation \((x^a) \in X\), there is an agent \( \beta \in A^1 \) such that

\[ x^\beta + \sum_{a \in A^0} \omega^a (\geq y^\beta) = 0. \]  

(9)

As previously mentioned, my analysis differs from that of Balasko et al. [2] and Wilson [15] in that the above irreducibility condition is imposed only on the full economy.

For \( x = (x_1, \ldots) \) define \( x_\gamma = (x_1, \ldots, x_\gamma, 0, \ldots) \), for \( \gamma \geq 1 \).

One immediate implication of (2) and (9) is that demand is insatiable; precisely, \( x \in X^a \) and \( x \leq \omega \) imply \( x + y (\geq y^a) x \) for some \( y \in R^\omega \). In fact, one may then infer from (4) that, for \( \alpha \in A \), there is a \( \gamma(\alpha) < \infty \) such that

\[ x \in X^a \text{ and } x \leq \omega \text{ imply } x + y(\gamma(\alpha)) (\geq y^a) x \text{, for some } y \in R^\omega. \]

In words, agent \( \alpha \)'s preferences for goods 1, \ldots, \( \gamma(\alpha) \) are insatiable. To establish Proposition 2, I strengthen this implication.

\(^9\) The potential problem being that an infinite sum of arbitrary upper hemi-continuous corresponds need not be upper hemi-continuous.
ASSUMPTION (local non-satiation). For \( x \in A \), there is a \( \gamma(x) < \infty \) such that, for \( \varepsilon > 0 \),

\[
x \in X^n \text{ and } x \leq \omega \text{ imply } x + \hat{y}_{\gamma(x)} (\geq \gamma(x)) x, \text{ for some } y \in R^{\infty} \text{ with }
\| \hat{y}_{\gamma(x)} \| < \varepsilon.
\]

(10)

A price system for the economy is a vector \( p = (p_1, \ldots) \), where \( p_i \) denotes the price of good \( i \geq 1 \). Given prices \( p \geq 0 \) and income \( I \geq 0 \), agent \( \alpha \)'s market demand correspondence is defined by

\[
d^\alpha(p, I) = \{ x \in X^n: p \cdot x \leq I \text{ and } p \cdot z > I \text{ for } z \in P^\alpha(x) \}.
\]

(11)

Following Debreu [6], I employ bounded (pseudo) demand correspondences in my equilibrium existence proofs of Propositions 2 and 3. These pseudo correspondences correspond to market demand when consumption sets are particular bounded subsets of \( R^{\infty}_{+} \). Specifically, for \( \kappa \geq 1 \),

\[
B_\kappa = \prod_{\gamma = 1}^\infty [0, \kappa \cdot \omega [.
\]

(12)

\[
d^\alpha_\kappa(p, I) = \{ x \in B_\kappa: p \cdot x \leq I \text{ and } p \cdot z > I \text{ for } z \in (P^\kappa(x) \cap B_\kappa) \}.
\]

(13)

Given demand and pseudo demand correspondences, I specify three types of price-allocation solutions for the economy.

A \( \kappa \)-solution consists of price system \( p \geq 0 \) and an allocation scheme \( (x^\alpha) \in X \) satisfying, for \( \alpha \in A \):

\[
p \cdot x^\alpha < \infty
\]

\[
p \cdot \omega^\kappa < \infty
\]

\[
x^\alpha \in d^\alpha(p, p \cdot x^\alpha).
\]

(14)

A transfer equilibrium consists of a price system \( p \geq 0 \) and an allocation scheme \( (x^\alpha) \in X \) satisfying, for \( \alpha \in A \): 11

\[
p \cdot x^\alpha < \infty
\]

\[
p \cdot \omega^\kappa < \infty
\]

\[
x^\alpha \in d^\alpha(p, p \cdot x^\alpha)
\]

\[
\sum_{\alpha \in A} x^\alpha = \omega.
\]

(15)

\[
As usual, local non-satiation follows from non-satiation when preferences are convex \( (y \in P^\alpha(x) \text{ and } 0 < \lambda \leq 1 \text{ imply } \lambda y + (1 - \lambda) x \in P^\alpha(x)) \), as assumed in [15].

11 The fact that I may guarantee finite expenditures and finite wealth in equilibrium follows from the discounting induced by the continuity of infinite dimensional preferences (4). It remains to be seen if such discounting can be dropped and the general existence of equilibria be meaningfully established with finite or infinite personal expenditure and wealth.
A competitive equilibrium is a transfer equilibrium where, for \( a \in A \),

\[
p \cdot x^* = p \cdot \omega^*.
\] (16)

The \( \kappa \)-solutions are merely employed in an intermediate step in the proof of the existence of transfer equilibria in Proposition 2. The transfer equilibria are those that can be supported through competitive markets after a (lump-sum) redistribution of income. As usual, competitive equilibria are those that can attain through competitive markets without governmental intervention.

3. Compactness of Transfer Allocations

**Proposition 1** (compactness of certain collections of \( \kappa \)-solutions). For any collection of positive scalars \((\Theta^a)_{a \in A}\), any sequence of pairs \(\{(p(\kappa)), (x^a(\kappa))\}\) satisfying

\[
p(\kappa) \text{ and } (x^a(\kappa)) \text{ are a } \kappa \text{-solution, for } \kappa \geq 1,
\]

\[
p(\kappa) \cdot x^a(\kappa) = 1, \quad \text{for } \kappa \geq 1 \text{ and some designated } a^* \in A
\]

\[
p(\kappa) \cdot x^a(\kappa) \geq \Theta^a p(\kappa) \cdot \omega^*, \quad \text{for } \kappa \geq 1 \text{ and } a \in A
\]

\[
\left[ \sum_{a \in A} x^a(\kappa) \right] \Rightarrow \omega \quad \text{as } \kappa \to \infty
\]

has a limit point, i.e., \((p(\kappa)), (x^a(\kappa))) \Rightarrow (p, (x^a))\), for some subsequence. Furthermore, the limit \((p, (x^a))\) is a transfer equilibrium and \(p(\kappa) \cdot x^a(\kappa) \Rightarrow p \cdot x^a\), for \( a \in A \).

Before establishing Proposition 1, I state a corollary that is useful in proving Propositions 3 and 6. The corollary follows immediately from the simple observation that transfer equilibria constitute \( \kappa \)-solutions for any \( \kappa \geq 1 \).

**Corollary 1** (compactness of transfer equilibria). For any collection of positive scalars \((\Theta^a)_{a \in A}\); the transfer equilibria \((p, (x^a))\) satisfying

\[
p \cdot x^a = 1, \quad \text{for some designated } a^* \in A
\]

\[
p \cdot x^a \geq \Theta^a p \cdot \omega^* \quad \text{for } a \in A
\]

constitute a (sequentially)\(^{12}\) compact subset of \(R_+^A \times \mathcal{X}\).

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\(^{12}\) Since the product topology on \(R^A\) is metrizable (consider the metric \(d(x, y) = \sum_{i=1}^Af_i \min(x_i - y_i, 1)\)), compactness is equivalent to sequential compactness Munkres [13, p. 181].
Proof of Proposition 1. Consider any such sequence of $\kappa$-solutions. The Cantor diagonalization process insures that, for some subsequence $\{(p(\kappa), (x^\kappa(\kappa)))\}$,

for $\alpha, \beta \in A$ and $\gamma \geq 1$, either $\{x^\alpha(\kappa)\}$ converges to a finite limit or it diverges to infinity as $\kappa \to \infty$ \hspace{1cm} (23)

for $\alpha, \beta \in A$ such that $p(\kappa) \cdot x^\beta(\kappa) > 0$ for large $\kappa$, either $\{p(\kappa) \cdot x^\alpha(\kappa)/p(\kappa) \cdot x^\beta(\kappa)\}$ converges to a finite limit or it diverges to infinity as $\kappa \to \infty$ \hspace{1cm} (24)

for $\gamma \geq 1$, either $\{p_\gamma(\kappa)\}$ converges to a finite limit or it diverges to infinity as $\kappa \to \infty$. \hspace{1cm} (25)

I now establish that each of the sequences in (23)–(25) actually converges. Condition (20) implies $\lim \sup_{\kappa \to \infty} x^\kappa(\kappa) \leq \omega, \omega < \infty$. In particular, $x^\kappa(\kappa)$ does not diverge to infinity as $\kappa \to \infty$. Condition (23) then implies, for $\alpha \in A$,

$$x^\alpha(\kappa) \Rightarrow x^\alpha \text{ (pointwise convergence), for some } x^\alpha \geq 0. \hspace{1cm} (26)$$

Equation (1) applied to each allocation $(x^\kappa(\kappa)) \in X$, together with (7), (20), and (26), yields\(^{13}\)

$$\sum_{\alpha \in A} x^\alpha = \omega. \hspace{1cm} (27)$$

To guarantee that the sequence in (24) converges, I establish that, for every $\alpha$ and $\beta \in A$ such that $p(\kappa) \cdot x^\beta(\kappa) > 0$ for large $\kappa$,

$$\{p(\kappa) \cdot x^\alpha(\kappa)/p(\kappa) \cdot x^\beta(\kappa)\} \text{ does not converge to } 0. \hspace{1cm} (28)$$

Condition (24) implies that $A$ may be partitioned into two sets

$$A^0 = \{\alpha \in A: \{p(\kappa) \cdot x^\alpha(\kappa)/p(\kappa) \cdot x^\beta(\kappa)\} \text{ converges to } 0\}$$

\(^{13}\) Without assumption (7) one could merely conclude $\sum_{\alpha \in A} x^\alpha \leq \omega$ since, in general, $\sum_{\kappa \in A} x^\kappa \leq \liminf_{\kappa \to \infty} \sum_{\kappa \in A} x^\kappa(\kappa) = \omega$. This is the reason I impose (7) and restrict $x_\gamma^\kappa = 0$ unless agent $\alpha$ desires good $\gamma$ in the definition (1) of the space of allocations $X$. 
and

\[ A^1 = \{ \alpha \in A: \frac{p(\kappa) \cdot x^\alpha(\kappa)}{p(\kappa) \cdot x^\beta(\kappa)} \text{ either converges to a positive limit or diverges to infinity} \}. \]

Clearly \( \beta \in A^1 \) and—assuming (28) is violated—\( A^0 \) is also non-empty. Hence, irreducibility (9) may be applied to \( A^0, A^1 \) and the limiting allocation \( (x^\beta) \)—since it satisfies (27)—to yield an agent \( \beta' \in A^1 \) satisfying

\[ x^\beta' + \sum_{\alpha \in A^2} \omega^\alpha (\sup x^\beta) x^\beta'. \]

Continuity (4) then implies that there is a \( \lambda < 1 \), integer \( \gamma \geq 1 \), and a finite subset \( A_0^0 \subseteq A^0 \) satisfying \( \lambda \hat{x}_\gamma^\beta' + \sum_{\alpha \in A^2} \omega^\alpha (\sup x^\beta) x^\beta' \). Again, (4) yields

\[ \lambda \hat{x}_\gamma^\beta' + \sum_{\alpha \in A^2} \omega^\alpha (\sup x^\beta) x^\beta(\kappa), \quad \text{for large } \kappa. \tag{29} \]

Clearly, definition (12) implies \( \lambda \hat{x}_\gamma^\beta' + \sum_{\alpha \in A^2} \omega^\alpha \in B_\delta \), for large \( \kappa \). Therefore, \( x^\beta(\kappa) \in d^\beta(\kappa, p(\kappa), p(\kappa) \cdot x^\beta(\kappa)) \), (13) and (29) imply

\[ \lambda p(\kappa) \cdot \hat{x}_\gamma^\beta' + \sum_{\alpha \in A^2} p(\kappa) \cdot \omega^\alpha > p(\kappa) \cdot x^\beta(\kappa). \tag{30} \]

Fix a \( \lambda' \in (\lambda, 1) \). Convergence (26) implies \( \lambda \hat{x}_\gamma^\beta' \leq \lambda' x^\beta(\kappa) \), for large \( \kappa \). With (30) this implies

\[ \sum_{\alpha \in A^2} p(\kappa) \cdot \omega^\alpha > p(\kappa) \cdot x^\beta(\kappa) - \frac{\lambda'}{1 - \lambda'} p(\kappa) \cdot \hat{x}_\gamma^\beta' \]

\[ \geq (1 - \lambda') p(\kappa) \cdot x^\beta(\kappa). \tag{31} \]

Therefore,

\[
\lim_{\kappa \to \infty} \frac{p(\kappa) \cdot x^\beta(\kappa)}{p(\kappa) \cdot x^\beta(\kappa)} \leq \frac{1}{1 - \lambda'} \lim_{\kappa \to \infty} \sum_{\alpha \in A^2} \frac{p(\kappa) \cdot \omega^\alpha}{p(\kappa) \cdot x^\beta(\kappa)}
\]

\[ \leq \frac{1}{1 - \lambda'} \sum_{\alpha \in A^2} \frac{1}{\Theta^\alpha \lim_{\kappa \to \infty} \frac{p(\kappa) \cdot x^\alpha(\kappa)}{p(\kappa) \cdot x^\beta(\kappa)}} = 0, \]

where the first inequality follows from (31), the second from (19) and the finiteness of \( A_0^0 \) and the equality from the definition of \( A^0 \) (recall \( A_0^0 \subseteq A^0 \)). But, this contradicts \( \beta' \in A^1 \).
The above paragraph establishes (28). Clearly (28) implies that the sequences in (24) converge to a positive limit; to show that such sequences do not diverge to infinity, reverse the indices $\alpha$ and $\beta$ and apply (28). Therefore, for $\alpha \in A$, applying positive convergence (24) to $\alpha$ and $\beta$ equal to the $\alpha^*$ designated in (18) yields

$$\{p(\kappa) \cdot x^*(\kappa)\} \text{ converges to a positive limit.}$$  \hfill (32)

Assumption (8) and (27) imply that, for any $\gamma \geqslant 1$, there is an $\alpha \in A$ such that $x^*_\alpha \geqslant 0$. But, (32) implies that $\{p(\kappa) \cdot x^*_\alpha(\kappa)\}_{\kappa \geqslant 1}$ is bounded. Hence, convergence (26) implies that $\{p(\kappa)\}_{\kappa \geqslant 1}$ is bounded. By (25), this guarantees

$$p(\kappa) \Rightarrow p, \quad \text{for some } p \geqslant 0. \hfill (33)$$

I now establish, for $\alpha \in A$,

$$x^* \in d^*(p, [\lim_{\kappa \to \infty} p(\kappa) \cdot x^*(\kappa)]). \hfill (34)$$

Since $p(\kappa), x^*(\kappa) \geqslant 0$; convergence (26) and (33) imply $p \cdot x^* \leqslant \lim_{\kappa \to \infty} p(\kappa) \cdot x^*(\kappa)$—that is, $x^*$ is affordable given prices $p$ and income $[\lim_{\kappa \to \infty} p(\kappa) \cdot x^*(\kappa)]$. Hence, assuming (34) fails, (11) implies that there is some alternative $y \in X^*$ that is both preferred $y > x^*$ and affordable $p \cdot y \leqslant [\lim_{\kappa \to \infty} p(\kappa) \cdot x^*(\kappa)]$. Continuity (4) and positive convergence (32) then imply that for some $\lambda < 1$ and integer $\gamma \geqslant 1$,

$$\lambda \hat{y}, (> )^* x^* \hfill (35)$$

and

$$\lambda p \cdot \hat{y}, < [\lim_{\kappa \to \infty} p(\kappa) \cdot x^*(\kappa)]. \hfill (36)$$

But, for large $\kappa$, continuity (4) and (35) imply

$$\lambda \hat{y}, (> )^* x^*(\kappa),$$

convergence (33) and (36) imply

$$p(\kappa) \cdot (\lambda \hat{y}, ) < p(\kappa) \cdot x^*(\kappa),$$

and definition (12) implies

$$\lambda \hat{y}, \in B^*_\kappa.$$
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But, the above three expressions contradict definition (13) of \( x^a(\kappa) \in d^a(p(\kappa), p(\kappa) \cdot x^a(\kappa)) \).

The above paragraph establishes (34). Local non-satiation (10) and (34) then imply \( p \cdot x^a = \lim_{\kappa \to \infty} p(\kappa) \cdot x^a(\kappa) \), for \( \alpha \in A \). Properties (27), (32), (34), and the above equality imply that \((p, x)\) is a transfer equilibrium (see (15)) provided that \( p \cdot \omega^a < \infty \), for \( \alpha \in A \). But, hypothesis (19) and convergence (33) imply

\[
p \cdot \omega^a \leq \lim_{\kappa \to \infty} \inf p(\kappa) \cdot \omega^a \leq \frac{1}{\Theta^a} \lim_{\kappa \to \infty} \inf p(\kappa) \cdot x^a(\kappa) = \frac{p \cdot x^a}{\Theta^a} < \infty.
\]

4. EXISTENCE OF COMPETITIVE EQUILIBRIA

In this section, I establish the general existence of competitive equilibria for dynamic economies. Following Balasko et al. [2] and Wilson [15], I find an equilibrium for the full (infinite) economy to be a limit point of equilibria from a sequence of truncated (finite) sub-economies. (The existence of this limit follows from Proposition 1.) In order to employ classical results guaranteeing the existence of equilibria for each of the truncated sub-economies, Balasko et al. and Wilson require that each sub-economy be irreducible.

In this paper, I eliminate the need for sub-economy irreducibility assumptions—which, as usual, are imposed to guarantee that each agent has a positive income. My procedure is to apply the above truncation method to first establish existence in the special case of agents having positive endowments of all goods—the positiveness of endowments combined with local non-satiation (10) insures the irreducibility of the sub-economies. I then establish existence for my general model by perturbing endowments to be positive, applying my special existence result for economies with positive endowments, and then shrinking the perturbations to zero (fast enough) so that the limit of the perturbed-economy equilibria is a competitive equilibrium for the original (unperturbed) economy.

**Proposition 2** (existence when endowments are positive). If

\[
\omega^a \gg 0
\]

(37)

for all \( \alpha \in A \) and, for some \( \alpha^* \in A \) and \( \varepsilon > 0 \),

\[
\omega^a \gg \varepsilon \cdot \omega;
\]

(38)

then the economy has a competitive equilibrium.
I establish Proposition 2 employing only well-known results for finite economies.

**Proof.** For convenience index agents by $A = \{1, 2, \ldots\}$ in such a way that the $\gamma(\alpha)$ defined in (10) are ordered—that is,

$$\gamma(\alpha) \leq \gamma(\alpha + 1) \quad \text{for} \quad \alpha = 1, 2, \ldots \quad (39)$$

Fix any $\kappa = 1, 2, \ldots$. Form a $\kappa$-truncated economy consisting of agents $1, \ldots, \kappa$ and goods $1, \ldots, \tilde{\gamma}$—where $\tilde{\gamma} \equiv \gamma(\kappa)$. For $\alpha \leq \kappa$, agent $\alpha$ has the consumption space $\tilde{B}_\kappa = \prod_{\gamma = 1}^{\tilde{\gamma}} [0, \kappa \omega_\gamma]$ with a preference relation over bundles $(x_1, \ldots, x_{\tilde{\gamma}})$ and $(y_1, \ldots, y_{\tilde{\gamma}}) \in \tilde{B}_\kappa$ defined by

$$(x_1, \ldots, x_{\tilde{\gamma}}) \text{ is preferred to } (y_1, \ldots, y_{\tilde{\gamma}})$$

$$\iff (x_1, \ldots, x_\gamma, \kappa \omega_{\gamma+1}, \ldots) \succ (y_1, \ldots, y_\gamma, \kappa \omega_{\gamma+1}, \ldots), \quad (40)$$

where $(\succ)^\alpha$ is the preference relation of agent $\alpha \in A$ in the original economy.

I extend every price system $p = (p_1, \ldots, p_{\tilde{\gamma}})$ for the truncated economy to the system $(p_1, \ldots, p_\gamma, 0, \ldots)$ for the original economy. By free disposal (6) and (40), demand $d^\alpha(p, \cdot)$ in the truncated economy is related to pseudo demand $d^\alpha_{\kappa}(\cdot)$—see (12) and (13)—in the original economy by

$$d^\alpha(p, I) = \{(x_1, \ldots, x_{\tilde{\gamma}}) \in \tilde{B}_\kappa:\$$

$$(x_1, \ldots, x_\gamma, \kappa \omega_{\gamma+1}, \ldots) \in d^\alpha_{\kappa}((p_1, \ldots, p_\gamma, 0, \ldots), I)\} \quad (41)$$

Finally, define truncated endowments $\tilde{\omega}^\alpha = (\omega_1^\alpha, \ldots, \omega_{\tilde{\gamma}}^\alpha)$ for $\alpha \leq \kappa$.

One may readily verify that the $\kappa$-truncated economy inherits all of the properties (2)–(10) of the original economy described Section 2; in particular, irreducibility (9) follows from (37) and the local insatiability of preferences for goods, $1, \ldots, \tilde{\gamma}$ implied by (10), (39), and (40). Therefore, I may invoke classical results (e.g., Gale and Mas-Colell [7, 8], Mas-Colell [10], or McKenzie [12]) to establish the existence of a competitive equilibrium for the truncated economy.

Let $\tilde{p} \in \mathbb{R}_{+}^\tilde{\gamma}$ ($p \neq 0$) denote the (truncated) equilibrium prices and $\tilde{x}^\alpha \in \tilde{B}_\kappa$ the consumption bundle of agent $\alpha \leq \kappa$. That is,

$$\tilde{x}^\alpha \in d^\alpha(\tilde{p}, \tilde{p} \cdot \tilde{\omega}^\alpha) \quad \text{for agents} \quad \alpha = 1, \ldots, \kappa \quad (42)$$

$$\sum_{\alpha = 1}^{\kappa} \tilde{x}^\alpha_{\gamma} = \sum_{\alpha = 1}^{\kappa} \tilde{\omega}^\alpha_{\gamma} \quad \text{for goods} \quad \gamma = 1, \ldots, \tilde{\gamma}. \quad (43)$$
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I define a \( \kappa \)-solution \( (p(\kappa), (x^\kappa(\kappa))) \) for the full economy from the above competitive equilibrium for the \( \kappa \)-truncated economy by

\[
p(\kappa) = (\bar{p}_1, \ldots, \bar{p}_\gamma, 0, \ldots)
\]

\[
x^\alpha(\kappa) = (x_1^\alpha, \ldots, x_{\gamma}^\alpha, \kappa \omega_{\kappa+1}, \ldots), \quad \text{for} \quad \alpha = 1, \ldots, \kappa
\]

\[
x^\alpha(\kappa) = \kappa \omega, \quad \text{for} \quad \alpha = \kappa + 1, \ldots
\]

Technically, I restrict \( x^\alpha(\kappa) = 0 \) unless agent \( \alpha \) desires good \( \gamma \) in (45) and (46) so that \( (x^\alpha(\kappa)) \) is a legitimate allocation scheme—see (1). One may verify from (5), (6), (12)–(14), and (41)–(46) that \( (p(\kappa), (x^\alpha(\kappa))) \) is indeed a \( \kappa \)-solution.

I now allow \( \kappa \) to vary. (37), (42) and \( p(\kappa) \neq 0 \) imply \( p(\kappa) \cdot x^{\prime}(\kappa) > 0 \) for every \( \kappa \). Hence, I may innocuously normalize prices by \( p \cdot x^{\prime}(\kappa) = 1 \)—see (18). The construction of every \( (p(\kappa), (x^\alpha(\kappa))) \)—see especially (42)–(46)—with (7) implies that the sequence of \( \kappa \)-solutions \( \{(p(\kappa), (x^\alpha(\kappa)))\} \) also satisfies (19) and (20) of proposition 1 for parameters \( \Theta^\alpha = 1, \) for \( \alpha \in A \). Hence, there is a subsequence that converges to a transfer equilibrium \( (p, (x^\alpha)) \)—that is, as \( \kappa \to \infty \),

\[
p(\kappa) \Rightarrow p \quad \text{and} \quad (x^\alpha(\kappa)) \Rightarrow (x^\alpha).
\]

Proposition 1 also implies, for \( \alpha \in A \),

\[
p(\kappa) \cdot x^\alpha(\kappa) \Rightarrow p \cdot x^\alpha.
\]

Local non-satiation (10), (42), and (44) imply

\[
p(\kappa) \cdot x^\alpha(\kappa) = p(\kappa) \cdot \omega^\alpha
\]

for \( \alpha \leq \kappa \). Convergence (47) implies \( p \cdot \omega^\alpha \leq \lim \inf_{\kappa \to \infty} p(\kappa) \cdot \omega^\alpha \); hence, (48) and (49) imply, for \( \alpha \in A \),

\[
p \cdot \omega^\alpha \leq p \cdot x^\alpha.
\]

Applying (15)—specifically, \( p \cdot x^\alpha < \infty \)—and (38) and (50) to the \( x^\alpha \) of \( (38) \), yields

\[
p \cdot \omega < \infty.
\]

Since \( \sum_{\alpha \in A} x^\alpha = \sum_{\alpha \in A} \omega^\alpha = \omega \), (51) implies that both \( \sum_{\alpha \in A} p \cdot x^\alpha \) and \( \sum_{\alpha \in A} p \cdot \omega^\alpha \) are convergent and equal to \( p \cdot \omega \). Hence,

\[
0 = \sum_{\alpha \in A} p \cdot x^\alpha - \sum_{\alpha \in A} p \cdot \omega^\alpha = \sum_{\alpha \in A} (p \cdot x^\alpha - p \cdot \omega^\alpha),
\]
which, by (50), implies \( p \cdot x^a = p \cdot \omega^a \), for every \( a \in A \). Since it satisfies (16), the transfer equilibrium \((p, (x^a))\) is also a competitive equilibrium for the full economy. Q.E.D.

The following existence results strengthen Theorems 1 and 2 of Wilson [15] in that weaker regularity conditions and no sub-economy irreducibility conditions are imposed.

**Proposition 3** (general existence of equilibria with minimal transfers). There is a transfer equilibrium satisfying, for each \( a \in A \):

\[
p \cdot x^a \geq p \cdot \omega^a
\]

(52)

\[
p \cdot x^a = p \cdot \omega^a \quad \text{if} \quad \omega_i^a > 0 \text{ for only a finite number of goods } \gamma.
\]

(53)

**Proposition 4** (existence of competitive equilibria in special cases). If either

for each \( a \in A \), \( \omega_i^a > 0 \) for only a finite number of goods \( \gamma \),

(54)

or there is a finite subset of agents \( A' \subseteq A \) and \( \varepsilon > 0 \) satisfying

\[
\sum_{a \in A'} \omega^a \geq \varepsilon \cdot \omega,
\]

(55)

then a competitive equilibrium exists.

Wilson [15] provides a counter-example demonstrating that if both (54) and (55) are violated, then the economy need not have competitive equilibria—only transfer equilibria as described in Proposition 3.

**Proof of Proposition 4** (given Proposition 3). Consider the transfer equilibrium \((p, (x^a))\) generated by Proposition 3.

If (54) holds, then (16) and (53) assert that \((p, (x^a))\) is a competitive equilibrium. If (55) holds, then (15) implies \( p \cdot \omega \leq (1/\varepsilon) \sum_{a \in A'} p \cdot \omega^a < \infty \), since \( A' \) is finite. As in the previous proof, \( p \cdot \omega \leq \infty \) and (52) imply \( p \cdot x^a = p \cdot \omega^a \), for \( a \in A \); hence, by (16), \((p, (x^a))\) is a competitive equilibrium. Q.E.D.

**Proof of Proposition 3.** Let \( P^{1/2} \subseteq R^\infty_+ \) denote the collection of price systems corresponding to transfer equilibria satisfying (21) and (22) with parameters \( \Theta^a = 1/2 \), for \( a \in A \). Corollary 1 implies that \( P^{1/2} \) is compact; hence, bounded. Let \( \tilde{p} \in R^\infty_+ \) be an upper bound; that is, \( p \preceq \tilde{p} \) for \( p \in P^{1/2} \).

(56)

Choose \( p^* \in R^\infty_+ \) to satisfy

\[
\tilde{p} \cdot p^* < \infty.
\]

(57)
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Choose an array \( (\varepsilon^a) \) of consumption vectors—\( \varepsilon^a \in X^a \), for \( a \in A \)—to satisfy
\[
\sum_{a \in A} \varepsilon^a = p^*.
\] (58)

Assumption (8) insures that \( p^* \) and the \( \varepsilon^a \) may be chosen so that the \( \varepsilon^a \) are small enough to satisfy
\[
0 \leq \varepsilon^a \leq \omega^a, \quad \text{for} \quad a \in A.
\] (59)

Let \( A = \{1, 2, \ldots\} \). For \( \kappa = 4, 5, \ldots; \) form the \( \kappa \)-perturbed economy by redistributing endowments from \( (\omega^a) \) to \( (\omega^a(\kappa)) \), where
\[
\omega^a(\kappa) = (1 - \kappa^{-1})\omega^a + \kappa^{-1}(2^{-\gamma^a}p^* - \varepsilon^a), \quad \text{for} \quad a \neq \kappa \quad (60)
\]
\[
\omega^a(\kappa) = (1 - \kappa^{-1})\omega^a + \kappa^{-1}(2^{-\gamma^a}p^* - \varepsilon^a) + \kappa^{-1}\omega^a, \quad \text{for} \quad a = \kappa. \quad (61)
\]

Indeed, \( (\omega^a(\kappa)) \) is merely a redistribution of resources inasmuch as (58), (60), and (61) imply
\[
\sum_{a \in A} \omega^a(\kappa) = \sum_{a \in A} \omega^a. \quad (62)
\]

Clearly, \( p^* \geq 0 \) and (59)–(61) imply that \( (\omega^a(\kappa)) \) distributes positive endowments (37) while (59) and (61) imply that (38) is satisfied for \( a^* = \kappa \) and \( \varepsilon = \omega^a \). Hence, Proposition 2 establishes the existence of a competitive equilibrium \( (p(\kappa), (x^a(\kappa))) \) for the \( \kappa \)-perturbed economy. In particular, (16) yields for \( a \in A, \)
\[
p(\kappa) \cdot x^a(\kappa) = p(\kappa) \cdot \omega^a(\kappa). \quad (63)
\]

Consider the sequence \( \{(p(\kappa), (x^a(\kappa)))\} \). Equation (62) implies that the competitive equilibria of the perturbed economies constitute transfer equilibria of the original (unperturbed) economy. By standard arguments, (9) implies \( p(\kappa) \cdot \omega^a(\kappa) > 0 \), for \( a \in A \); hence, (21) is an innocuous normalization of prices \( p(\kappa) \). By inspection, (59)–(61) imply \( \omega^a(\kappa) \geq (1/2)\omega^a \), for \( a \in A \) and \( \kappa \geq 4 \). Hence, (63) guarantees that (22) is also satisfied given parameters \( \Theta = 1/2 \), for \( a \in A \). Therefore, Corollary 1 implies that, for some subsequences of equilibria \( \{(p(\kappa), (x^a(\kappa)))\}, \)
\[
(p(\kappa), (x^a(\kappa))) \Rightarrow (p, (x^a)) \quad (64)
\]

and, for \( a \in A, \) Proposition 1 implies
\[
p(\kappa) \cdot x^a(\kappa) \Rightarrow p \cdot x^a, \quad (65)
\]

where \( (p, (x^a)) \) is some transfer equilibrium of the original economy.
Since every $p(\kappa) \in B^{1/2}$, (56) implies $p(\kappa) \leq \bar{p}$. Equations (57) and (58) then imply $|p(\kappa) \cdot (2^{-s}p^* - \varepsilon^*| \leq \bar{p} \cdot p^* < \infty$. In particular, 

$$\kappa^{-1}p(\kappa) \cdot (2^{-s}p^* - \varepsilon^*) = 0, \quad \text{as} \quad \kappa \to \infty.$$ 

Equation (60) then implies, for $\alpha \in A$,

$$\liminf_{\kappa \to \infty} p(\kappa) \cdot \omega^\alpha(\kappa)$$

$$= \liminf_{\kappa \to \infty} (1 - \kappa^{-1}) p(\kappa) \cdot \omega^\alpha + \lim_{\kappa \to \infty} \kappa^{-1}p(\kappa) \cdot (2^{-s}p^* - \varepsilon^*)$$

$$= \liminf_{\kappa \to \infty} p(\kappa) \cdot \omega^\alpha.$$

Therefore, by (63) and (65),

$$p \cdot x^\alpha = \liminf_{\kappa \to \infty} p(\kappa) \cdot \omega^\alpha,$$

This equality yields (52) and (53) since, in general, convergence (64) implies $p \cdot \omega^\alpha \leq \liminf_{\kappa \to \infty} p(\kappa) \cdot \omega^\alpha$ and, if $\omega^\alpha > 0$ for only a finite number of $\gamma$, (64) implies $p \cdot \omega^\alpha = \liminf_{\kappa \to \infty} p(\kappa) \cdot \omega^\alpha$. Q.E.D.

5. Necessity of Non-standard Assumptions

Example 1 demonstrates the need for a regularity restriction on consumption spaces of infinitely lived agents in addition to standard closeness and convexity conditions. Specifically, I present an economy satisfying all the hypotheses of Section 2 with the single exception that the consumption set $X^\alpha$ of one of the infinitely lived agents is not $R^\infty_+$. As asserted in (2)—but rather a particular closed and convex subset of $R^\infty_+$. (The set $X^\alpha$ also satisfies the free disposal condition: $x \in X^\alpha$ and $x \geq y$ imply $y \in X^\alpha$.) Nevertheless, the economy exhibits no equilibria with non-negative transfers as described in Proposition 3. One should therefore view the requirement of consumption spaces to consist of all non-negative bundles not as a mere convenience but the embodiment of an essential non-standard regularity condition on infinitely lived agents.

In example 2, I demonstrate the necessity of (7), which limits the number of agents desiring any particular good to be finite. Specifically, I present an economy of finitely lived agents satisfying all the hypotheses of Section 2 with the single exception that all agents desire one particular good. Nevertheless, the economy has no competitive equilibria as described in Proposition 4.
Although assumption (7) is quite innocuous for dynamic overlapping-generations economies, it may be difficult to incorporate into a general framework that accommodates atomless agents since the latter typically allow every agent to desire every good. 

**Example 1** (necessity of a regularity condition like $X^\infty = R_+^\infty$). This example requires only two agents, $A = \{0, 1\}$. Agent 0 has the consumption set $X^0 = \{ x \in R_+^\infty : x_\gamma + 2^{\gamma-3} \cdot (x_\gamma - 1) \geq 0 \text{ for } \gamma = 3, 4, ... \}$, preferences represented by $u^0(x) = x_1$, and the endowment $\omega^0 = (2, 2, ...)$. Agent 1 has the consumption set $X^1 = R_+^\infty$, preferences represented by $u^1(x) = \sum_{\gamma>1} 4^{-\gamma} \cdot \min\{x_\gamma, 5\}$, and the endowment $\omega^1 = (2, 2, ...)$. 

Assume there is an equilibrium with non-negative transfers, $(p \cdot x^0 - p \cdot \omega^0) \geq 0$, for $x \in A$. Monotonicity implies $p \geq 0$. Market clearing, $\sum_{\gamma \in A} x^0_\gamma = \sum_{\gamma \in A} \omega^0_\gamma$, and $x^0_\gamma \in X^0$ imply $x^0_2 \geq 1 \cdot p \geq 0$, $x^0_2 \geq 1$, the definition of $X^0$, and utility maximization by agent 0 imply $(x^0_2, x^0_1, x^0_0, ...) = (1, 0, 0, ...)$. Market clearing, $x^0_2 + x^1_2 = (4, 4, 4, ...)$, then implies $(x^2_2, x^1_2, x^0_2, ...) = (3, 4, 4, ...)$. In particular, $(0, 0, 0, ...) \ll (x^2_2, x^1_2, x^0_2, ...) \ll (5, 5, 5, ...)$. Utility maximization by agent 1 imply $(p_2, p_3, ...) = p_2 \cdot (1, 4^{-1}, ...)$. 

A contradiction is now evident in that agent 0 is not maximizing $u^0(\cdot)$ by choosing 

$$x^0 = (x^0_2, 1, 0, 0, 0, ...)$$

since the alternative 

$$y = \left( x^0_2 + \frac{p_2}{2p_1}, 0, 1, 2, 2^2, ... \right)$$

is preferred $(y_1 > x^0_1)$, in the consumption set $X^0$, and affordable; $(p_2, p_3, ...) = p_2 \cdot (1, 4^{-1}, ...)$. Moreover, 

$$p \cdot y = p_1 x^0_2 + p_2 \cdot 2^{-1} + \sum_{\gamma=3}^{\infty} p_\gamma \cdot 2^{\gamma-3} = p_1 x^0_2 + p_2 \cdot (2^{-1} + 2^{-1}) = p \cdot x^0$$

Q.E.D.

**Example 2** (necessity of restricting the number of agents desiring a single good). Index goods by $\{0, 1, ...\}$ and agents by $\{1, 2, ...\}$. Agent $a \geq 1$ has $X^a = R_+^\infty$, preferences represented by $u^a(x) = x_0 + x_a$, and the endowment defined by $\omega^a_\gamma = 1$ for $\gamma = a - 1$ and $x$ with $\omega^a_\gamma = 0$ elsewhere. (Note: all agents desire good 0.) 

Assume there is a competitive equilibrium. Monotonicity implies $p \geq 0$. 

\[ p > 0 \text{ and utility maximization imply } x_\alpha^* = 0 \text{ unless either } y = 0 \text{ or } y = \alpha. \]

Thus, market clearing conditions simplify to
\[ \sum_{\alpha \geq 1} x_\alpha^* = \omega_\alpha \cdot \frac{1}{\omega_\alpha} = 1 \quad (67) \]
for good 0 and, for good \( y \geq 1,\)
\[ x_y^* = \omega_y + \omega_y^{y+1} = 2. \quad (68) \]

Equation (68) allows the budget constraint of agent \( \alpha > 1, \)
\[ p_0 x_0^* + p_\alpha x_\alpha^* = p_{\alpha-1} + p_\alpha, \]
to be solved for
\[ x_0^* = \frac{(p_{\alpha-1} - p_\alpha)}{p_0}. \quad (69) \]

Obviously, the non-negativity of consumption with (69) implies that the price sequence \( p_0, p_1, \ldots \) is non-increasing. In particular, \( p \leq p_0 \cdot (1, 1, \ldots). \)

In fact, I can show that the price sequence is constant since, if, for some \( \alpha \geq 1, \) \( \alpha < p_{\alpha-1}; \) then \( p_{\alpha-1} \leq p_0 \) implies \( p_\alpha < p_0; \) which, by utility maximization of agent 0, implies \( x_0^* = 0; \) which, by (69), implies \( p_\alpha = p_{\alpha-1}; \) which contradicts \( p_\alpha < p_{\alpha-1}. \) Therefore, \( p = p_0 \cdot (1, 1, \ldots). \)

But, \( p = p_0 \cdot (1, 1, \ldots) \) and (69) imply \( x_0^* = 0 \) for all agents \( \alpha; \) which violates (67).

Q.E.D.

6. Existence of Taxation Equilibria

In this section, I establish first the lack of equilibria with positive income (monetary) transfers and second the abundance of equilibria with negative transfers (taxes). As usual, an allocation scheme \((x^*) \in X\) is said to be weakly Pareto optimal if there are no alternative schemes \((y^*) \in X\) satisfying \( \sum_{\alpha \in A} y^* = \sum_{\alpha \in A} x^* \) and \( y^* > x^* \) for \( \alpha \in A. \)

**Proposition 5** (lack of equilibria with positive transfers). *If the initial endowment sequence \((\omega^*) \in X\) is weakly Pareto optimal, then there are no equilibria with positive transfers \((p \cdot x^* - p \cdot \omega^* > 0), \) for \( \alpha \in A. \)

**Proof.** Suppose \((p, (x^*))\) is such an equilibrium. By the local insatiability of preferences (10), \( p \cdot x^* > p \cdot \omega^* \) implies \( x^* > \omega^*. \) But this, along with the market clearing constraint \( \sum_{\alpha \in A} x^* = \omega = \sum_{\alpha \in A} \omega^*, \) implies that \((\omega^*)\) is not weakly Pareto optimal.

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14 If one were to consider transitive preferences (so that a weak relation \((\geq)^*\) could be deduced), then the concept of *strong Pareto optimality* is well defined. Given this, one could show that if the initial endowment sequence \((\omega^*) \in X\) is strongly Pareto optimal, then there are no equilibria with all non-negative transfers \((p \cdot x^* - p \cdot \omega^*) \geq 0\) and at least one positive transfer \((p \cdot x^* - p \cdot \omega^*) > 0).
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PROPOSITION 6 (abundance of taxation equilibria). If each agent $\alpha$ has an endowment of only a finite number of goods (i.e., $\omega^\alpha_\gamma > 0$ for only a finite number of $\gamma$), then corresponding to any collection of scalars ($\Theta^\alpha$) satisfying $0 \leq \Theta^\alpha < 1$ for $\alpha \in \mathcal{A}$, there is a transfer equilibrium $(p, x^\alpha)$ with $\Theta^\alpha$ being the average tax rate imposed on agent $\alpha$; that is,

$$p \cdot x^\alpha = (1 - \Theta^\alpha) p \cdot \omega^\alpha > 0 \quad \text{for} \quad \alpha \in \mathcal{A}.$$ 

Note that the abundance of taxation equilibria—like the existence of competitive equilibria—is only guaranteed for finitely endowed agents.

Proof. Let $\mathcal{A} = \{1, 2, \ldots\}$. Consider any $\kappa = 1, 2, \ldots$. Alter the economy by giving the endowments $\omega^\alpha(\kappa) = (1 - \Theta^\kappa) \omega^\alpha$ to agents $\alpha < \kappa$, $\omega^\alpha(\kappa) = \omega^\alpha + \sum_{\alpha = 1}^{\kappa-1} \Theta^\alpha \omega^\alpha$ to agent $\alpha = \kappa$, and $\omega^\alpha(\kappa) = \omega^\alpha$ to agents $\alpha > \kappa$.

Clearly, this economy satisfies the assumptions of Proposition 4. Hence, there is a competitive equilibrium $(p(\kappa), (x^\alpha(\kappa)))$ for the altered economy. In particular, $p(\kappa) \cdot x^\alpha(\kappa) = p(\kappa) \cdot \omega^\alpha(\kappa) = (1 - \Theta^\kappa) p(\kappa) \cdot \omega^\alpha$, for $\alpha < \kappa$.

Note that $(p(\kappa), (x^\alpha(\kappa)))$ constitutes a transfer equilibrium for the original economy since I merely redistributed existing endowments; that is, $\sum_{\alpha \geq 1} \omega^\alpha(\kappa) = \sum_{\alpha \geq 1} \omega^\alpha$.

It is a simple matter to verify that the sequence of equilibria $\{(p(\kappa), (x^\alpha(\kappa)))\}$ satisfies the hypotheses of corollary 1; hence, the sequence has a limit point $(p, (x^\alpha))$ that is also a transfer equilibrium. Furthermore, Proposition 1 implies $p(\kappa) \cdot x^\alpha(\kappa) \Rightarrow p \cdot x^\alpha$ as $\kappa \Rightarrow \infty$. Hence, for each $\alpha \in \mathcal{A}$,

$$p \cdot x^\alpha = \lim_{\kappa \Rightarrow \infty} p(\kappa) \cdot x^\alpha(\kappa) = (1 - \Theta^\kappa) \lim_{\kappa \Rightarrow \infty} p(\kappa) \cdot \omega^\alpha = (1 - \Theta^\kappa) p \cdot \omega^\alpha,$$

since $\omega^\alpha_\gamma > 0$ for only a finite number of goods $\gamma$. Q.E.D.

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