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Eliminating sunspot effects in overlapping generations models

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ABSTRACT

We reappraise the robustness of sunspot effects in overlapping-generations models. Azariadis's well-known example economies have stationary, deterministic fundamentals (preferences, technologies, and endowments), yet sunspots affect multiple equilibria. And those equilibria are robust to common perturbations of underlying fundamentals. However, we find that judiciously splitting each of Azariadis's goods into close substitutes eliminates sunspot effects, and the remaining equilibrium is unique and deterministic. Forecasting thus becomes conclusive, and there is no longer a role for government policy to stabilize excess volatility.

1. Introduction

We reappraise the robustness of sunspot effects in overlapping-generations models. (That is a first step toward determining whether the existence of sunspots is generic in some suitable topological space or whether the non-existence of sunspots is generic, which in turn determines whether there is a role for government policy to stabilize excess volatility.) Well-known examples of sunspot effects by Azariadis (1981) and by Spear et al. (1990) are robust in the sense that there are sunspot equilibria for an open set of economies in a topological space where the number of goods per period is *fixed*, and where the difference between alternative preference relations is measured by the uniform C^2-norm difference in utility functions. However, we find the set of economies with sunspot equilibria is no longer open (sunspot effects are no longer robust) if the topological space is expanded so that, say, an economy with *one* good per period is considered nearby an economy with *two* close substitutes each period (Burke, 2009).

This paper is an example of a forthcoming analysis of the genericity of sunspot equilibria. Here, we just construct one perturbation that eliminates sunspot effects for one of Azariadis's example economies (Azariadis, 1981). That establishes the non-robustness of sunspot effects, but says nothing about any alternative perturbations that maintain sunspot effects. Forthcoming analysis formally defines the topological space of economies to include the union over n of a sequence of standard metric spaces of economies with n goods in each period. (The pseudo metric distance between an economy with m goods per period and an economy with $n \geq m$ goods is defined as the distance between the economies after one of the goods in the first economy is split into $(n - m + 1)$ perfect substitutes, so that now both economies have the same number of goods.) Forthcoming analysis then formulates results about residual sets of economies, and considers various well-known example economies. In particular, some of the economies of Spear et al. (1990) can be perturbed (by judiciously splitting one of the n goods in each period into two close substitutes) to eliminate sunspot effects.

2. Fundamentals and sunspot equilibrium

This section extends the well-known stationary overlapping-generations sunspot model of Azariadis (1981) to allow two goods each time period. To avoid the non-stationary position of an initial generation, extend time into the infinite past as well as the infinite future.

At the beginning of time period t ($t = 0, \pm 1, \pm 2, \ldots$), Mr. t is born, then lives in period t (youth) and $t + 1$ (retirement). Outside money is the only asset; assume an imperishable supply

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of 1. Consumption goods are perishable; good type 1 is available in the first half of a time period, and type 2 in the second half.

Mr. t generates utility $u(c_{t+1}, n_t)$ from the two types of retirement consumption $c_{t+1} = (c_{t+1,1}, c_{t+1,2}) \in \Re^2_+$ and two types of youthful labor supplied $n_t = (n_{t1}, n_{t2}) \in \Re^2_+$. The consumption good is produced at constant returns to scale using labor, with 1 labor unit producing 1 consumption unit. Endow the young with 1/2 divisible unit of each type of labor.

Under rational expectations, consumers know future prices may depend on sunspots. To fix the stochastic sunspot process, let Ω_t be a finite set of conceivable sunspot types realized at time t, and $\Omega^t = \prod_{-\infty}^{t} \Omega^t$ be the set of conceivable histories up through time t. Assume a common probability belief $\pi(s_{t+1}|s^t) > 0$ of each sunspot type $s_{t+1} \in \Omega_{t+1}$ conditional on each history $s^t \in \Omega^t$.

Each period's sunspot type is revealed to everyone before production and trade occurs. Hence, Mr. t's production y_t is a function of history up through time t, and consumption c_{t+1} is a function of history through $t + 1$. Expected utility conditional on history $s^t \in \Omega^t$ is thus

$$E\left[u(c_{t+1}, y_t)|s^t\right] := \sum_{s_{t+1}\in\Omega_{t+1}} \pi(s_{t+1}|s^t)\, u(c_{t+1}(s_{t+1}, s^t), y_t(s^t)). \tag{1}$$

There, pairing sunspot $s_{t+1} \in \Omega_{t+1}$ at $t + 1$ with history $s^t \in \Omega^t$ through t forms history $s^{t+1} = (s_{t+1}, s^t) \in \Omega^{t+1}$ through $t + 1$.

A *rational expectations equilibrium* for utility $u(c_{t+1}, n_t)$ is a time path of goods prices $p_t : \Omega^t \to \Re^2_+$, of outputs $y_t : \Omega^t \to \Re^2_+$, and of consumptions $c_t : \Omega^t \to \Re^2_+$ that satisfy, for each time period $t = 0, \pm 1, \ldots,$

(R1) For each history $s^t \in \Omega^t$, output $y_t(s^t) \in \Re^2_+$ and the consumption function $c_{t+1}(\cdot, s^t) : \Omega_{t+1} \to \Re^2_+$ solve

$$\max\ E\left[u(c_{t+1}, y_t)|s^t\right]$$
$$\text{s.t. } p_{t+1}(s_{t+1}, s^t) \cdot c_{t+1}(s_{t+1}, s^t) = p_t(s^t) \cdot y_t(s^t),\quad \forall s_{t+1} \in \Omega_{t+1}$$
$$y_t(s^t) \in [0, 1/2]^2$$

(R2) (Goods market clearing) $c_{t+1} = y_{t+1}$

(R3) (Money market clearing) $p_t \cdot y_t = 1$.

There, p_t are time t goods prices in terms of money. The constraint $y_t(s^t) \in [0, 1/2]^2$ ensures labor does not exceed the 1/2 unit endowment of each type of labor.

A "rational expectations" equilibrium is a *perfect foresight* equilibrium if every function p_t, y_t, c_t is constant (consumers know the future with certainty); otherwise, it is a *sunspot* equilibrium.

Assumption. Ω_t contains a single type for time periods before one, and Ω_t contains at least two types after one.

Because Ω_t is a singleton for time periods before one, all equilibria initially exhibit perfect foresight. Equilibrium analysis is thus about whether sunspots can *begin* to matter, not about whether sunspots can continue to matter.

3. Perturbing away sunspot effects

This section perturbs away sunspot effects for a representative of Azariadis's example economies (Azariadis, 1981). Those examples have one aggregate good available each time period, so specialize the definitions from Section 2.

The *aggregated economy* consists of a one unit labor endowment, and a separable bivariate utility function

$$v(c_{t+1}, n_t) := -\frac{1}{3}(c_{t+1})^{-3} - \frac{1}{2}\bar{y}^{-5}(n_t)^2, \quad \text{for parameter } \bar{y} = 2/3. \tag{2}$$

Observation 1. *The aggregated economy (2) has a single stationary perfect foresight equilibrium,*[1] *with consumption and production $c_t = y_t = \bar{y}$ and prices $p_t = \bar{y}^{-1}$. And that is the only perfect foresight equilibrium such that $y_0 = \bar{y}$.*

There, $\bar{y}$ is the parameter from the definition of utility (2).

To see Observation 1, compute auxiliary functions

$$U(c) := c\frac{\partial v}{\partial c}(c, n) = c^{-3} \quad \text{and}$$
$$G(n) := -n\frac{\partial v}{\partial n}(c, n) = \bar{y}^{-5}n^2 \tag{3}$$

as in Azariadis (1981, Eq. 2). Hence, first-order conditions for constrained utility maximization (R1) reduce perfect foresight equilibrium conditions to

$$U(y_{t+1}) \geq G(y_t) \quad \text{and} \quad y_t \in (0, 1], \quad \text{with equality if } y_t < 1. \tag{4}$$

Evidently, the only stationary solution $y_{t+1} = y_t$ to that difference equation (4) is $y_t = \bar{y}$. And $\{\bar{y}\}$ is the only solution path $\{y_t\}$ such that $y_0 = \bar{y}$. □

Observation 2. *The aggregated economy (2) has multiple sunspot equilibria such that $y_0 = \bar{y}$.*

To see Observation 2, note the auxiliary functions (3) satisfy Azariadis's sufficient conditions for the existence of sunspot equilibria (Azariadis, 1981, Section II): future consumption and current leisure are local gross complements ($U'(\bar{y}) < 0$) and the equilibrium difference equation (4) is locally stable ($|U'(\bar{y})| > |G'(\bar{y})|$). Hence, multiple output paths $\{y_t\}$ are consistent with sunspot equilibria. For example, one path varies with period 1 sunspot types $\alpha, \beta \in \Omega_1$:

(S1) $y_t = \bar{y}$, for $t = 0, -1, -2, \ldots$
(S2) $y_1(\alpha) = 1/2$ and $y_1(\beta) = 3/4$, for $t = 1$
(S3) $y_t(\alpha) = f^{t-1}(y_1(\alpha))$ and $y_t(\beta) = f^{t-1}(y_1(\beta))$, for $t = 2, 3, 4, \ldots$

for the transition function $f(y_t) := \bar{y}^{5/3}y_t^{-2/3}$ over $y_t \in [0, 1]$. (Both $y_1(\alpha) = 1/2$ and $y_1(\beta) = 3/4$ are in the stable set $[\bar{y}^{5/2}, 1]$ of that transition function.) Define sunspot probabilities $\pi(\alpha)$ and $1 - \pi(\alpha)$ by the auxiliary functions (Azariadis, 1981, Eq. 11):

(S4) $\pi(\alpha)U(y_1(\alpha)) + (1 - \pi(\alpha))U(y_1(\beta)) = G(\bar{y})$.

Other examples allow long-run stationary sunspot cycles (Azariadis, 1981, Section II). □

Observation 1 says the perfect-foresight equilibrium is *conditionally determinate*: if the economy initially follows the stationary perfect foresight equilibrium path, then the future equilibrium stays on that path. Observation 2 overturns such "conditional determinacy", with a deterministic perfect foresight equilibrium path (S1) followed by a sunspot equilibrium path (S2) and (S3). Hence, we perturb the aggregated economy (2) from Observation 2 to eliminate sunspot effects and establish conditional determinacy for rational expectations equilibria. Following Burke (2009), the perturbation will disaggregate the single good into close substitutes.[2]

One underlying implicit assumption in any general-equilibrium model with a fixed number of goods available is that, if any one of the goods were disaggregated into two (or more) goods, then those disaggregated goods must be perfect substitutes. Thus Azariadis

[1] Our use of money as numéraire (R3) excludes the trivial perfect foresight equilibrium where money is worthless, and consumption and production are always zero.

[2] Burke (2009) uses disaggregation to perturb away indeterminate perfect foresight equilibria. The Burke (2009) class of models is different from ours, the perturbation is different, and the results are different: we perturb away all but one equilibrium path, while Burke (2009) restricts attention to local analysis and allows an infinite number of alternative paths to remain in a neighborhood of the historic equilibrium path.

implicitly assumes his aggregated model (2) is equivalent to a disaggregated version.

The *disaggregated economy* consists of an endowment of 1/2 unit in youth of each of the two types of labor, and a utility function

$$u(c_{t+1}, n_t) := -\frac{1}{3}(c_{t+1,1} + c_{t+1,2})^{-3} - \frac{1}{2\bar{y}^5}(n_{t1} + n_{t2})^2 \qquad (5)$$

over consumption $c_{t+1} = (c_{t+1,1}, c_{t+1,2}) \in \Re^2_+$ and labor $n_t = (n_{t1}, n_{t2}) \in \Re^2_+$.

There is an evident equivalence of the equilibria of the disaggregated economy (5) and the unique equilibrium (Observation 1) of the aggregated economy (2).

Observation 3. *The disaggregated economy (5) has an infinite number of stationary perfect foresight equilibria, indexed $c_t = y_t = (\gamma_1, \gamma_2)$ and $p_t = (1/\bar{y}, 1/\bar{y})$, for parameters $(\gamma_1, \gamma_2) \in [1/6, 1/2]^2$ such that $\gamma_1 + \gamma_2 = \bar{y}$.*

Observation 3 says prices are equal for the two perfect substitutes in the disaggregated economy (5), and the distribution of consumption between substitutes is indeterminate. Henceforth, fix any one of the stationary output solutions (γ_1, γ_2) from Observation 3. Without loss of generality, assume $\gamma_1 < 1/2$.

A *perturbed economy* has the same endowments as the disaggregated economy (5) but a different utility function (6).

Observation 4. *For sufficiently large k, the perturbed utility function*

$$u^k := u - \frac{1}{k}\psi,$$

for polynomial $\psi(c_{t+1}, n_t) := (n_{t2}^2 + 1)\, \|c_{t+1} - (\gamma_1, \gamma_2)\|^2 \qquad (6)$

is concave, increasing in c_{t+1}, and decreasing in n_t when restricted to feasible consumption $c_{t+1} \in [0, 1]^2$ and labor $n_t \in [0, 1/2]^2$. And u^k can be extended to have the latter three properties over all nonnegative consumption and labor.

In Observation 4, u is the disaggregated version (5) of the aggregated utility function (2).

Keeping endowments fixed, measure the metric distance $d(u^k, u)$ between perturbed utility functions (6) and the disaggregated utility function (5) by the C^2-norm difference in utility restricted to feasible consumption. In particular, Observation 4 implies $d(u^k, u) \to 0$. Hence, measure the distance $\tilde{d}(u^k, v)$ between perturbed utility functions (6) and the aggregated utility function (2) as $\tilde{d}(u^k, v) := d(u^k, u)$. Hence, $d(u^k, u) \to 0$ implies the sequence of perturbed economies (6) converges to the aggregated economy (2). Our main result is that each of those perturbations eliminate the sunspot effects from the aggregated economy (2).

Theorem. *Each perturbed economy (6) has a single stationary perfect foresight equilibrium, with consumption and production $c_t = y_t = (\gamma_1, \gamma_2)$ and prices $p_t = (\bar{y}^{-1}, \bar{y}^{-1})$. And that is the only rational expectations equilibrium such that $y_0 = (\gamma_1, \gamma_2)$ and $p_0 = (\bar{y}^{-1}, \bar{y}^{-1})$.*

The theorem says each perturbed economy lacks the sunspot equilibria that Observation 2 finds for the aggregated economy (2). The convergence of the sequence of perturbed economies to the disaggregated economy (5) thus proves those sunspot effects are fragile.

To prove the theorem, fix any perturbed economy u^k (6). The perfect foresight equilibrium $c_t = y_t = (\gamma_1, \gamma_2)$ and $p_t = (\bar{y}^{-1}, \bar{y}^{-1})$ from Observation 3 for the disaggregated economy (5) is also an equilibrium for the perturbed economy (6) because, at $c_t = y_t = (\gamma_1, \gamma_2)$, utility functions u and u^k have the same first derivatives (6), and so satisfy the same first-order conditions for utility maximization (R1).

To prove uniqueness for the perturbed economy, consider any alleged rational expectations equilibrium paths $\{p_t\}, \{y_t\}, \{c_t\}$ such that $y_0 = (\gamma_1, \gamma_2)$ and $p_0 = (\bar{y}^{-1}, \bar{y}^{-1})$.

Keep Mr. 0's retirement consumption c_1 on the equilibrium path, but consider changing youthful output from y_0 to $y_\varepsilon := y_0 + (\varepsilon, -\varepsilon)$, for ad hoc parameter $\varepsilon \geq 0$. $\gamma_1 \in [1/6, 1/2)$ and $\gamma_2 \in [1/6, 1/2]$ imply, for sufficiently small ε, output $y_\varepsilon \in [0, 1/2]^2$ and the consumption function $c_1(\cdot, s^0) : \Omega_1 \to \Re^2_+$ are feasible for Mr. 0's utility maximization problem (R1) at prices $p_0 = (\bar{y}^{-1}, \bar{y}^{-1})$. Hence, Mr. 0's solution at equilibrium y_0 implies a non-positive marginal utility

$$0 \geq \frac{\partial}{\partial \varepsilon} E\left[u^k(c_1, y_\varepsilon)|s^0\right]\Big|_{\varepsilon=0}.$$

Computing that derivative (1), (5) and (6) yields

$$0 \geq \frac{\partial}{\partial \varepsilon} E\left[u^k(c_1, y_\varepsilon)|s^0\right]\Big|_{\varepsilon=0} = \frac{2\gamma_2}{k} E\left[\|c_1 - (\gamma_1, \gamma_2)\|^2 \,\Big|\, s^0\right].$$

Hence, $\gamma_2 > 0$ implies $E\left[\|c_1 - (\gamma_1, \gamma_2)\|^2 \,\Big|\, s^0\right] = 0$, which implies $c_1 = (\gamma_1, \gamma_2)$ in every sunspot state $s_1 \in \Omega_1$ since conditional probability $\pi(s_1|s_0) > 0$. Hence, goods market clearing (R2) implies $y_1 = (\gamma_1, \gamma_2)$. Finally, $c_1 = (\gamma_1, \gamma_2)$ implies utility functions u and u^k have the same first derivatives (6), and so y_0 and c_1 can only maximize utility u^k (R1) if period 1 prices in vector p_1 are equal for the two substitute goods. Hence, money market clearing (R3) implies $p_1 = (\bar{y}^{-1}, \bar{y}^{-1})$.

Putting it all together, $y_0 = (\gamma_1, \gamma_2)$ and $p_0 = (\bar{y}^{-1}, \bar{y}^{-1})$ at time 0 imply $y_1 = (\gamma_1, \gamma_2)$ and $p_1 = (\bar{y}^{-1}, \bar{y}^{-1})$ at every sunspot history $s^1 \in \Omega^1$. Repeating that argument implies $y_t = (\gamma_1, \gamma_2)$ and $p_t = (\bar{y}^{-1}, \bar{y}^{-1})$ at every sunspot history $s^t \in \Omega^t$. Hence, that perfect foresight equilibrium is the only rational expectations equilibrium such that $y_0 = (\gamma_1, \gamma_2)$ and $p_0 = (\bar{y}^{-1}, \bar{y}^{-1})$ at time 0. □

References

Azariadis, C., 1981. Self-fulfilling prophecies. Journal of Economic Theory 25, 380–396.

Burke, J., 2009. Virtual determinacy in overlapping generations models. Econometrica 77, 235–247.

Spear, Stephen E., Srivastava, Sanjay, Woodford, Michael, 1990. Indeterminacy of stationary equilibrium in stochastic overlapping generations models. Journal of Economic Theory 50, 265–284.