Weird Dice

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Ordinary Dice

- These are the dice we’re familiar with
- 6 sides
- Labels 1 through 6
- Probabilities of rolling a 2 through 12
- Are there any other ways of labeling cubes (or any polyhedron) to give us the same probabilities?

Finding Our First Pair of Weird Dice

- $(x^{a1} + \ldots + x^{a6})(x^{b1} + \ldots + x^{b6}) = (x^6 + x^5 + x^4 + x^3 + x^2 + x)^2 = x^2(x + 1)^2(x^2 + x + 1)^2(x^2 - x + 1)^2$
- Now, let $P(x) = (x^{a1} + \ldots + x^{a6})$
- This means $P(x) = x^q(x + 1)^r(x^2 + x + 1)^t(x^2 - x + 1)^u$
- $P(1) = 1^{a1} + \ldots + 1^{a6} = 6$ and $P(1) = 1^q 2^r 3^t 1^u \rightarrow r = 1, t = 1$

Finding Our First Pair of Weird Dice

- $P(0) = 0^{a1} + \ldots + 0^{a6} = 0$ and $P(0) = 0^q 1^r 2^t 1^u = 0$
- But if $q = 0$, then we would have $P(0) = 1(1^r)(2^t)(1^u)$ which does not equal 0
- If $q = 2$, we have another contradiction
- Therefore $q = 1$
- Now we must only check the cases when $u = 0, 1, \text{or } 2$

Finding Our First Pair of Weird Dice

- If $u = 0$, $P(x) = x(x + 1)(x^2 + x + 1) = x^4 + 2x^3 + 2x^2 + x$. So the die labels are 4, 3, 3, 2, 2, 1.
- When $u = 1$, $P(x) = x(x + 1)(x^2 + x + 1)(x^2 - x + 1) = x^6 + x^5 + x^4 + x^3 + x^2 + x$. So the labels are 6, 5, 4, 3, 2, 1 – an ordinary die
- When $u = 2$, $P(x) = x(x + 1)(x^2 + x + 1)(x^2 - x + 1)^2 = x^8 + x^6 + x^5 + x^4 + x^3 + x$. Therefore the die labels are 8, 6, 5, 4, 3, 1.

Finding Our First Pair of Weird Dice

	8	6	5	4	3	1
4	12	10	9	8	7	5
3	11	9	8	7	6	4
3	11	9	8	7	6	4
2	10	8	7	6	5	3
2	10	8	7	6	5	3
1	9	7	6	5	4	2

$(\text{Weird Dice})^2$

- Let's see if we can do the same with an 18-sided die and a 2-sided die
- $x^2(x + 1)^2(x^2 + x + 1)^2(x^2 - x + 1)^2 = (x^{a1} + \ldots + x^{a18})(x^{b1} + x^{b2})$
- So, let $P(x) = x^{b1} + x^{b2} = x^q(x + 1)^r(x^2 + x + 1)^t(x^2 - x + 1)^u$
- $P(1) = 1^{b1} + 1^{b2} = 2$ and $P(1) = 1^q 2^r 3^t 1^u \rightarrow r = 1, t = 0$
- Evaluating P(0) two ways just as before we see find that $q = 1$ again
- So we only must check when $u = 0, 1,$ or 2

$(\text{Weird Dice})^2$

- If $u = 0$, $P(x) = x(x + 1) = x^2 + x$. So the die labels are 2, 1.
- When $u = 1$, $P(x) = x(x + 1)(x^2 - x + 1) = x^4 + x$. So the labels are 4, 1
- When $u = 2$, $P(x) = x(x + 1)(x^2 - x + 1)^2 = x^6 - x^5 + x^4 + x^3 - x^2 + x$. Therefore u can not equal 2

$(\text{Weird Dice})^2$

- Doing the same analysis for the 18-sided die, we learn $q = 1$, $t = 2$, and $r = 1$
- We only must check when $u = 0, 1,$ or 2
- If $u = 0$, $P(x) = x(x + 1)(x^2 + x +1)^2 = x^6 + 3x^5 + 5x^4 + 5x^3 + 3x^2 + x$. So the die labels are 6, 5, 4, 3, 2, 1.
- When $u = 1$, $P(x) = x(x + 1)(x^2 + x +1)(x^2 - x +1) = x^8 + 2x^7 + 3x^6 + 3x^5 + 3x^4 + 3x^3 + 2x^2 + x$. So the labels are 8, 7, 6, 5, 4, 3, 2, 1
- When $u = 2$, $P(x) = x(x + 1)(x^2 + x +1)^2 (x^2 - x +1)^2 = x^{10} + x^9 + 2x^8 + 2x^7 + 3x^6 + 3x^5 + 2x^4 + 2x^3 + x^2 + x$. Therefore the die labels are 10, 9, 8, 7, 6, 5, 4, 3, 2, 1

(Weird Dice)^2

	10	9	8	8	7	7	6	6	6	5	5	5	4	4	3	3	2	1
1	11	10	9	9	8	8	7	7	7	6	6	6	5	5	4	4	3	2
2	12	11	10	10	9	9	8	8	8	7	7	7	6	6	5	5	4	3

	8	7	7	6	6	6	5	5	5	4	4	4	3	3	3	2	2	1
1	9	8	8	7	7	7	6	6	6	5	5	5	4	4	4	3	3	2
4	12	11	11	10	10	10	9	9	9	8	8	8	7	7	7	6	6	5

Work Cited

- Gallian, Joseph. <u>Contemporary Abstract Algebra</u>. New York: Houghton Mifflin, 2005.