Use of models in Teaching Linear Algebra

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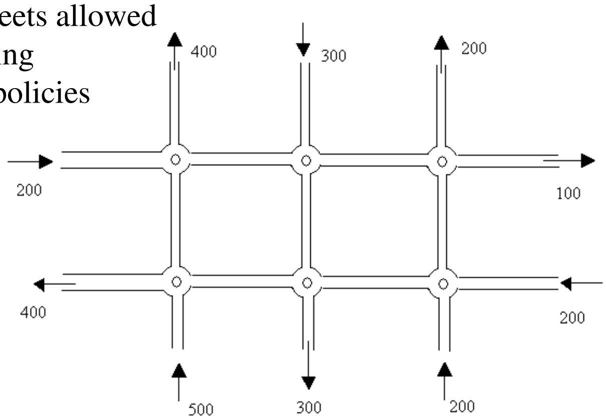
Traffic Problem

Introducing the problem setting and its assumptions

- Grid of road through two main financial blocks in a City
- Given number of vehicles passing through streets per hour
- Roundabouts redirect traffic
- No parking on streets allowed
- Interest in analyzing traffic diversion policies



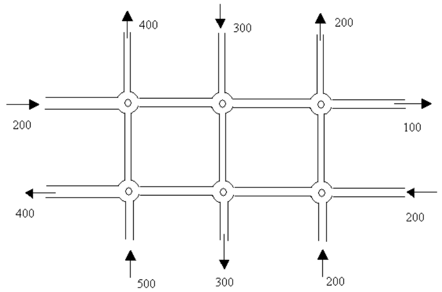
Traffic Problem

Questions to be answered by students:

1. Is it possible to close off one or more roads? If so which ones can be closed?

2. If we were to set minimum quantities of cars to circulate in a particular road (stretch between roundabouts) what would this amount be in order to maintain normal flow in the system?

3. Is it possible to consider a restriction of no more than 200 cars each hour in a particular road?



Research Questions

- Is it possible to introduce students to important concepts in linear algebra through the use of models?

- How to design teaching strategies based on a models and modeling perspective using APOS?

- Which aspects of students' mathematical knowledge can be retrieved and enhanced through the use of the above approach?

Preliminaries

- Students have difficulties formulating and solving systems of linear equations related to “real-life” problems.
- Students have difficulties relating equations to their graphical representation (and the graphical representation of the solution space).
- Some research has been developed on the use of APOS in designing and implementing activities to teach main concepts involved in understanding and solving systems of linear equations. (Manzanero 2007; Trigueros & Oktaç 2007)

Theoretical Framework

- Models and Modeling Approach
  - Model-eliciting activities.
  - Help students develop ideas in a meaningful realistic context.
- APOS
  - Learning Theory (Actions-Process-Object-Schema)
  - Successful in understanding of teaching in advanced mathematics.

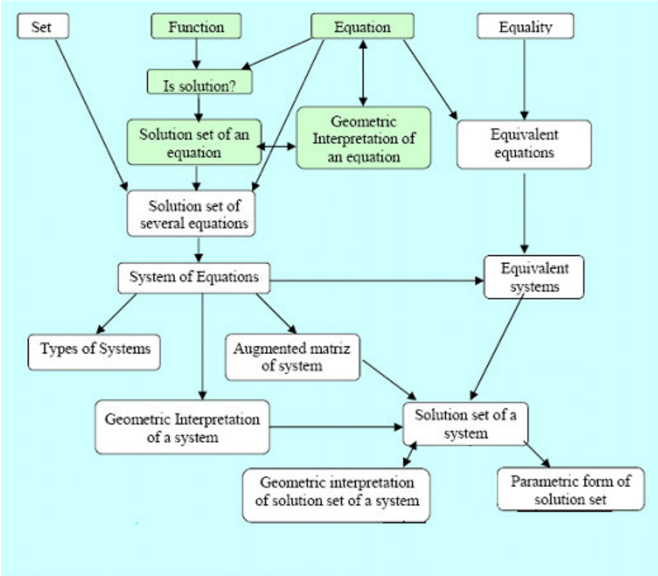
How modeling comes into play

- Provides realistic complex situations, where students engage in mathematical thinking.
- Conceptual tools are generated to arrive at a usable model. These tools are useful in decision making.
- Problems to model should satisfy certain conditions (Lesh 2005, English 2004)
- Rich context problems are used together with sessions where more controlled activities are introduced, that respond to conceptual needs that come from the modeling process.

Modeling from an APOS perspective

- Students use constructed schema for mathematics and for other domains.
- Actions over the components of these schema are interiorized in the processes of selection of variables, relations between them, and coordination of schema into the mathematization process of the problem and in turn to a model.
- The model becomes an object of study (model substitutes problem)
- As students act on the model, new schemas need to be constructed to answer questions.
- Actions and processes are developed to construct objects and schemas, which are in turn used to study the model.
- Spiral cycling until a goal is achieved.

Genetic Descomposition



RUMEC group

Methodology

- Linear Algebra at university level.
- Small private university.
- Preliminary experience with some groups
- Several modeling cycles.
- Students working in small groups.
- Observation and analysis of whole group discussion.

PLAN:

- Four groups (four different teachers), engineering, social sciences and economics undergraduate students.
- Analysis of students' production.
- Discussion between teachers and researchers.
- Study of evolution of schema and interaction between students is the focus of this study.

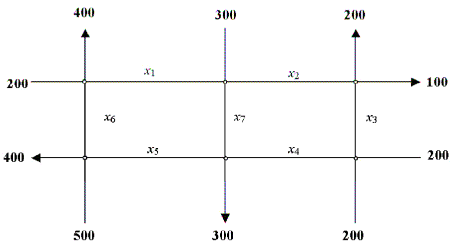
Cycles

1. Selecting and relating variables.
2. Student manipulation of the set of linear equations.
3. Matrix representation and its algebraic manipulation.
4. Answering specific questions and the graphical representation of the solution space.

Cycles

1. Selecting and relating variables

- Small group discussion, teacher visits groups
- Problems identifying variables
  - "The variables are: the cars, the streets"
  - Students miss the key word *number of (quantity)*
- Understanding flow of cars in the roundabouts as the characteristic that permit the setting of linear equations to represent the system (idea of a balanced system)



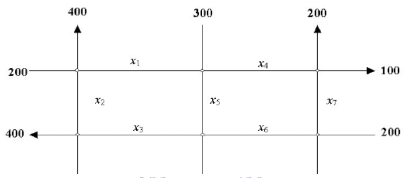
$$\begin{aligned}
200 + x_6 &= 400 + x_1 \\
x_1 + 300 &= x_2 + x_7 \\
x_2 + x_3 &= 200 + 100 \\
200 + 200 &= x_3 + x_4 \\
x_7 + x_4 &= x_5 + 300 \\
500 + x_5 &= 400 + x_6
\end{aligned}$$

Cycles

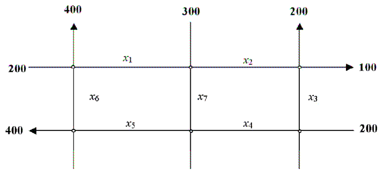
1. Selecting and relating variables

- Discussion about differences between groups in their choice of variable name impacting the system of linear equations.

- Equivalent systems, non uniqueness of equations.



$$
\begin{aligned}
x_2 + 200 &= 400 + x_1\\
500 + x_3 &= x_2 + 400\\
x_1 + 300 &= x_4 + x_5\\
x_5 + x_6 &= 300 + x_3\\
x_7 + x_4 &= 200 + 100\\
200 + 200 &= x_7 + x_6
\end{aligned}
$$



$$
\begin{aligned}
200 + x_6 &= 400 + x_1\\
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\end{aligned}
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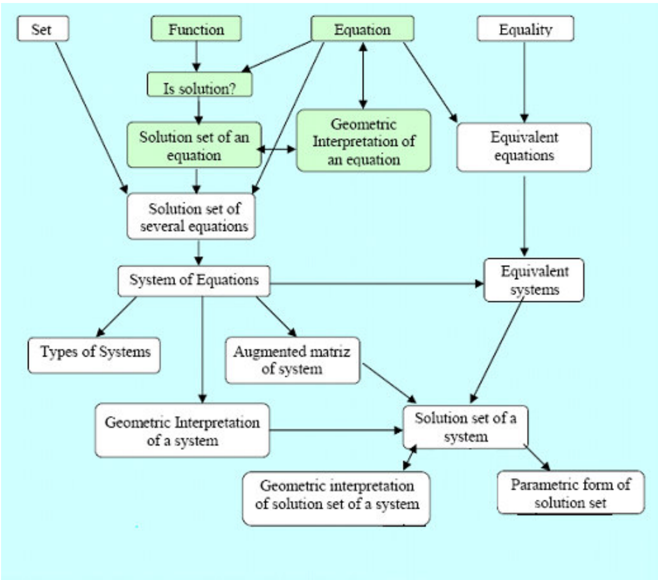
Genetic Descomposition



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Cycles

2. System Manipulation

- Different ways of manipulating the set of linear equations.

$$
\begin{array}{rrrrrrrcr}
x_1 & & & & & -x_6 & & = & -200 \\
x_1 & -x_2 & & & & & -x_7 & = & -300 \\
 & x_2 & +x_3 & & & & & = & 300 \\
 & & x_3 & +x_4 & & & & = & 400 \\
 & & & x_4 & -x_5 & & +x_7 & = & 300 \\
 & & & & x_5 & -x_6 & & = & -100
\end{array}
\qquad
\begin{array}{rrrrrrrcr}
-x_1 & & & & & +x_6 & & = & 200 \\
x_1 & -x_2 & & & & & -x_7 & = & -300 \\
 & x_2 & +x_3 & & & & & = & 300 \\
 & & -x_3 & -x_4 & & & & = & -400 \\
 & & & x_4 & -x_5 & & +x_7 & = & 300 \\
 & & & & x_5 & -x_6 & & = & -100
\end{array}
$$

- Substituting numbers (values) to look at plausible solutions.

$$
\begin{array}{rcl}
x_1 & = & 100 \\
x_6 & = & 200 + x_1 = 300 \\
x_2 & = & 100 \\
x_7 & = & -x_1 + x_2 + 300 \\
x_7 & = & 300 \\
\vdots & &
\end{array}
$$

- Problems with algebraic manipulation motivate the need for matrix representation and systematic algorithm to solve system of equations.

Cycles

3. Matrix representation and algebraic manipulation

- Actions on the manipulation of coefficients (reflecting questions)
- Action of setting the augmented system
- Row operations

$$
\left(\begin{array}{rrrrrrr|r}
-1 & 0 & 0 & 0 & 0 & 1 & 0 & -200 \\
1 & -1 & 0 & 0 & 0 & 0 & -1 & -300 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & -300 \\
0 & 0 & -1 & -1 & 0 & 1 & 0 & 400 \\
0 & 0 & 0 & 1 & -1 & 0 & 1 & 300 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & -100
\end{array}\right)
$$

Internalize the process of solving a system of equations via augmented matrix

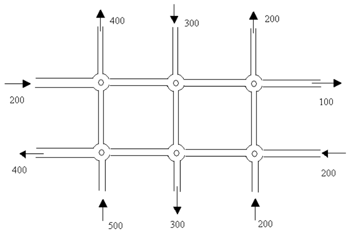
- Option of introducing Gauss-Jordan
- Comparing work with other groups (non-uniqueness of matrix)
- Reflection on the meaning of the different matrix and equivalent systems of equations.
- Not all representation as easy to manipulate.

Cycles

3. Matrix representation and algebraic manipulation

- Eliciting the use of adjacency matrix as an object to represent the road layout.

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}$$



- Generalizing and extrapolating objects to other problem settings

- Matrix $\leftrightarrow$ System of Equations $\leftrightarrow$ Road Layout

Cycles

4. Answering specific questions and graphical representation of solution space

- Obtaining solutions (in groups). Multiple solutions
- Activities or group discussion, to write the solution in terms of the parameters

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} -200 + t_1 \\ 100 + t_1 - t_2 \\ 200 - t_1 + t_2 \\ 200 + t_1 - t_2 \\ -100 + t_1 \\ t_1 \\ t_2 \end{pmatrix}$$

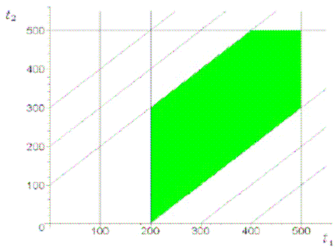
- Answering specific questions
- Comparing with other parameterizations

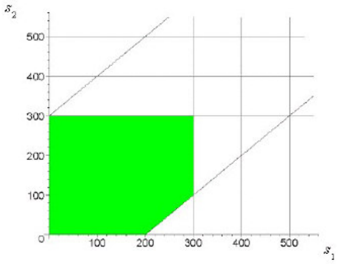
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ 300 - s_2 \\ 100 + s_2 \\ 100 + s_1 + s_2 \\ 200 + s_1 + s_2 \\ 300 - s_2 \end{pmatrix}$$

Cycles

4. Answering specific questions and graphical representation of solution space

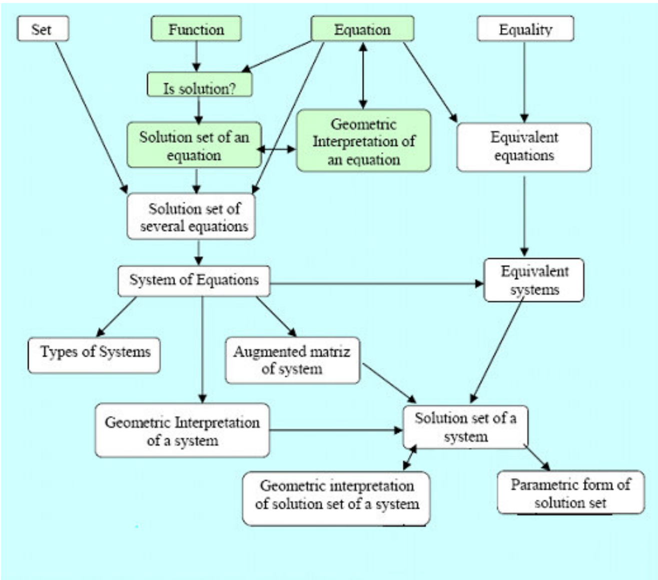
- Setting activities of graphing the solutions space for different parameterizations.
- Identifying the value range for the different problem variable





- Process of coordinating the value ranges and graphical interpretation to answer specific questions.

Genetic Descomposition



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Conclusions

- It is possible to work with the integrated theoretical framework
- The use of a genetic decomposition is a useful guide to design activities and introduce concepts.
- Theoretical frameworks complement each other and guide teacher's decisions
- Modeling provides opportunities for students to reflect on previous and current knowledge
- APOS theory used in the design of activities and the analysis of data.
- Teachers also develop conceptual tools (activities, assessment tools)
- Mathematics education theories can be extended to contexts different from those where they were originally posed.