# Do not open this exam until told to do so. 

## Pepperdine Math Day

November 15, 2014
Exam Instructions and Rules

1. Write the following information on your Scantron form:

Name in NAME box
Grade in Subject box
School name in Date box (and into Period box, if necessary)
On the back of your Scantron form, write this same information on the first line of the green shaded area.
2. This exam will last $\mathbf{9 0}$ minutes. It is a $\mathbf{4 5}$ question multiple choice exam. Each question is followed by answers marked A, B, C, D and E. Exactly one answer is correct for each problem. You will use the first 45 spots on the front page of the scantron form to record your answers. Your answer to the tie-breaker question should be written on the backside of the Scantron form below your name, etc. in the green shaded area. Your answer to this problem does not count toward your score-it will be used only for tiebreaking.
3. On this exam, there is no penalty for incorrect answers, so it is to your advantage to put an answer for each question, especially if you are able to eliminate one or more of the answers as incorrect. Credit will be given only for answers on your scantron form, not for any work written on the exam itself.
4. Use a number 2 pencil to mark your answer. Be sure to completely darken each of your penciled-in answers. Extra pencils are available from proctors.
5. There should be enough space between problems (or the backside of pages) to work your solutions. Credit is given only for answers on your Scantron answer form, not for any work written on the exam or scratch paper.
6. Figures are not necessarily drawn to scale.
7. While we certainly don't expect it, any sort of cheating will be dealt with at the discretion of the proctors, and will likely include at least disqualification.

1. A regular tetrahedron (triangle-base pyramid with all edges the same length) with edge length 2 has volume:
(a) $2 \sqrt{2} / 3$
(b) 3
(c) $2 \sqrt{3} / 3$
(d) $4 \sqrt{3}$
(e) $3 \sqrt{3}$

2. How many ways are there to place 8 different chess pieces on an $8 \times 8$ chessboard so that no two pieces are in the same row or column?
(a) $8^{8}$
(b) $(8!)^{2}$
(c) 64
(d) $8^{3}$
(e) $8^{64}$
3. Which of the following is equal to $\cos (4 x)$ ?
(a) $4 \cos x$
(b) $\cos ^{4} x$
(c) $2 \cos ^{4} x+1$
(d) $8 \cos ^{4} x-8 \cos ^{2} x+1$
(e) $\cos ^{4} x+4 \cos ^{3} x+6 \cos ^{2} x+4 \cos x+1$
4. An equilateral triangle of side length 2 has area:
(a) $\sqrt{2}$
(b) 3
(C) $\sqrt{3}$
(d) $4 / 3$
(e) $1+\sqrt{2}$
5. What is the number of the parking spot the car is in?
Luman mix
(a) 17
(b) 66
(c) 67
(d) 78
(e) 87
6. Amy, Bryan, and Christie are in rockets each traveling in straight lines. These three lines are perpendicular to each other. All three start at the position ( $0,0,0$ ). One hour later, Amy is at $(1,1,0)$, and Bryan is at $(1,-1,1)$. Which of the following could be where Christie is then?
(a) $(1,0,1)$
(b) $(0,-1,1)$
(c) $(1,-1,-2)$
(d) $(-1,1,1)$
(e) $(1,1,1)$
7. Let $f(x)$ be the function $1+a x+b x^{2}$ for which $f(1)=2$ and $f(2)=4$. What is $3 a+5 b$ ?
(a) -8
(b) -4
(C) 4
(d) 8
(e) It cannot be determined from the given information.
8. The Fibonacci sequence is defined as $F_{1}=1, F_{2}=1, F_{n+2}=F_{n}+F_{n+1}$. What is the final digit of $F_{600}$ ?
(a) 0
(b) 1
(c) 2
(d) 3
(e) 7
9. There are 1000 bank accounts, numbered 1 to 1000 . Bank account 1 has $\$ 1$. Bank account 2 has $\$ 2$, and so on, so that bank account $n$ has $\$ n$. How many dollars are in all of the bank accounts combined?
(a) $\$ 100,000$
(b) $\$ 123,456$
(C) $\$ 500,500$
(d) $\$ 1,000,000$ (e) $\$ 1,001,000$
10. If the areas of a square with side $s$ and circle with diameter $d$ are the same, what is the ratio $s / d$ ?
(a) $\pi / 4$
(b) $\pi^{2} / 4$
(C) $\sqrt{\pi} / 2$
(d) $\pi / 2$
(e) $\pi^{2} / 2$
11. Which of the following are the real solutions to $x^{4}-2 x^{2}-1=0$ ?
(a) $1 \pm \sqrt{2}$
(b) $+\sqrt{1 \pm \sqrt{2}}$
(C) $\pm \sqrt{1+\sqrt{2}}$
(d) $\pm \sqrt{1-\sqrt{2}}$
(e) $\pm \sqrt{1 \pm \sqrt{2}}$
12. If $\frac{a}{b}=\frac{2}{3}$ and $\frac{c}{d}=\frac{4}{5}$, what is the value of $\frac{a+c}{b+d}$ ?
(a) $6 / 8$
(b) $3 / 4$
(c) $22 / 15$
(d) $6 / 15$
(e) It is not uniquely determined by the given information.
13. Initially an urn contains 50 blue marbles and 50 red marbles. Repeatedly, three marbles are randomly removed from the urn and replaced with two marbles from a pile outside the urn as follows:

| Marbles removed | Replaced with |
| :--- | :--- |
| 3 blue | 1 blue, 1 red |
| 2 blue, 1 red | 1 blue, 1 red |
| 1 blue, 2 red | 2 red |
| 3 red | 1 blue, 1 red |

For example, if three marbles are randomly selected from the urn and if 2 are blue and 1 is red, then these three are removed and 1 blue and 1 red marble are put back into the urn. Which of the following sets of marbles could be the contents of the urn after repeated applications of this procedure?
(a) 2 blue marbles
(b) 2 red marbles
(c) 1 blue marble
(d) 1 red marble
(e) 1 blue marble, 1 red marble
14. The addition below is incorrect. We want to change just one digit to make the expression correct. There is more than one way to accomplish this. What is the value of the largest digit that can be changed to make the addition correct?

$$
\begin{array}{r}
641 \\
852 \\
+973 \\
\hline 2456
\end{array}
$$

(a) 4
(b) 5
(c) 6
(d) 7
(e) 8
15. If $1<a \leq b$, what is the smallest possible value of $\log _{a}(a / b)+\log _{b}(b / a)$ ?
(a) -1
(b) $1 / e$
(c) 1
(d) 3
((e)) It has no minimum value.
16. If $a / b$ is the value that is halfway between $20 / 50$ and $42 / 49$ written in simplest (i.e. reduced) form, what is $a+b$ ?
(a) 15
(b) 31
(c) 32
(d) 57
(e) 114
17. An equilateral triangle is inscribed inside of a circle. A second circle is inscribed inside of the triangle. What is the smallest possible ratio between the radius of the smaller circle to the radius of the larger circle?
(b) $1: 2$
(b) $1: \sqrt{2}$
(c) $\sqrt{2}: 1$
(d) $1: \sqrt{3}$
(e) $\sqrt{3}: 1$
18. The post office has come out with a new set of stamps featuring famous mathematicians (it's about time!). There are stamps worth 4 cents, 10 cents and 29 cents. What is the largest value of postage that you cannot get using only stamps of these kinds. (You are allowed to use as many of each type of stamp as you wish.)
(a) 28 cents
(b) 31 cents
(c) 35 cents
(d) 38 cents
(e) 97 cents
19. Suppose you have two large bottles. Bottle 1 contains 4 cups of Apple Juice and Bottle 2 contains 4 cups of Orange Juice. You pour 1 cup of Apple Juice from Bottle 1 and add it to Bottle 2. You thoroughly mix together the contents of Bottle 2, then pour 1 cup from Bottle 2 back into Bottle 1. What is the ratio of the amount of Apple Juice in Bottle 1 to Orange Juice in Bottle 2?
(a) $1: 1$
(b) $2: 1$
(c) $1: 2$
(d) $4: 1$
(e) $1: 4$
20. The following four statements, and only these, are found on a card:

On this card, exactly one statement is false.
On this card, exactly two statements are false.
On this card, exactly three statements are false.
On this card, exactly four statements are false.
How many of the statements must be false?
(a) None
(b) One
(c) Two
((d)) Three
(e) All four
21. Mike has a solid wooden cube with whole number dimensions. He paints the entire surface of the cube red. Then, with slices parallel to the faces of the cube, Tim cuts the cube into $1 \times 1 \times 1$ cubes. The case when the cube is $3 \times 3 \times 3$ is shown at right.
Let:
$x$ be the number of the small cubes completely free of paint $y$ the number of the small cubes painted red on only one side $z$ the number of the small cubes painted red on two sides.

(Of course there will be 8 small cubes painted red on three sides.) If $y$ is twice as large as $x$, what was Mike's original cube size?
(a) $4 \times 4 \times 4$
(b) $5 \times 5 \times 5$
(c) $6 \times 6 \times 6$
(d) $7 \times 7 \times 7$
(e) $8 \times 8 \times 8$
22. Let $L$ and $S$ be operators where $a L b$ selects the larger of the two numbers and $a S b$ selects the smaller of the two numbers. For example, $3 L 4=4$ and $3 S 4=3$. Of course, $a L a=a$ and $a S a=a$. How many of the following rules are correct?

$$
\begin{aligned}
a L b & =b L a \\
a L(b L c) & =(a L b) L c \\
a S(b S c) & =(a S b) L(a S c) \\
a L(b L c) & =(a L b) S(a L c)
\end{aligned}
$$

(a) 0
(b) 1
(c) 2
(d) 3
(e) 4
23. A particle moves through the first quadrant as follows. During the first minute it moves from the origin $(0,0)$ to $(1,0)$, then to $(1,1)$, then to $(0,1)$, and so on. Thereafter it continues to follow the directions indicated in the figure, going back and forth between the positive $x$ and $y$ axes, moving one unit of distance parallel to an axis in each minute. At what point will the particle be after 2014 minutes?
(a) $(20,14)$
(b) $(43,16)$
(c) $(43,17)$
(d) $(44,10)$
(e) $(44,20)$

24. The graph at right shows a portion of the curve defined by the quartic polynomial

$$
P(x)=x^{4}+a x^{3}+b x^{2}+c x+d
$$

Which of the following is the smallest?
(a) $P(-1)$
(b) The product of the zeroes of $P$
(c) The product of the non-real zeroes of $P$
(d) The sum of the coefficients of $P$
(e) The sum of the real zeroes of $P$

25. Find the value of the expression

$$
\frac{1}{\log _{2} 100!}+\frac{1}{\log _{3} 100!}+\frac{1}{\log _{4} 100!}+\cdots+\frac{1}{\log _{100} 100!}
$$

(a) $1 / 100$
(b) $1 / 10$
(C) 1
(d) 2
(e) 10
26. Four distinct lines lie in a plane, and exactly two of them are parallel. Which of the following could be the number of points where two (or more) of the lines intersect?
(a) 3 only
(b) 5 only
(c) 3 or 4 only
(d) 3 or 5 only
(e) 3 or 4 or 5
27. How many whole numbers between 100 and 400 contain the digit 2 ?
(a) 100
(b) 120
(C) 138
(d) 140
(e) 148
28. For what real numbers $p$ is $p x^{2}+p x-1 \leq 0$ for all $x$ ?
(a) $p \leq-4$
(b) $-\sqrt{2} \leq p \leq 0$
(c) $p \leq 0$
(d) $\sqrt{2}-1 \leq p \leq 0$
(e) $-4 \leq p \leq 0$
29. Let $f$ be a function satisfying $f(x y)=\frac{f(x)}{y}$ for all positive real numbers $x$ and $y$. If $f(500)=3$, what is $f(600)$ ?
(a) 1
(b) 2
(C) $5 / 2$
(d) 3
(e) $18 / 5$
30. The sum of seven integers is -1 . What is the maximum number of the seven integers that could be larger than 13 ?
(a) 1
(b) 4
(c) 5
(d) 6
(e) 7
31. Which one of the following sets of three shapes cannot be rearranged into a square?

(a)

(b)

(c)

(d)
(e) All four could be rearranged into squares.
32. Suppose that all three zeros of the polynomial $P(x)=x^{3}+a x^{2}+b x+c$ are real, and that the mean of the zeros, the product of the zeros, and the sum of the four coefficients $(1, a, b, c)$ are all equal. If the $y$-intercept of the graph of $y=P(x)$ is 2 , what is $b$ ?
(a) -11
(b) -9
(c) -7
(d) 0
(e) 1
33. Let $f(x)=\frac{1}{x+1}$. Find $f(f(2)+f(0)+f(1)+f(3))$.
(a) $\frac{12}{37}$
(b) $\frac{30}{91}$
(c) $\frac{61}{30}$
(d) $\frac{30}{61}$
(e) None of (a) - (e)
34. For all integers $0<n \leq 2014$, let

$$
\begin{aligned}
& \qquad a_{n}=\left\{\begin{aligned}
11 & \text { if } n \text { is divisible by } 13 \text { and } 14, \text { but not } 11 \\
13 & \text { if } n \text { is divisible by } 11 \text { and } 14, \text { but not } 13 \\
14 & \text { if } n \text { is divisible by } 11 \text { and } 13, \text { but not } 14 \\
0 & \text { otherwise }
\end{aligned}\right. \\
& \text { Calculate } \sum_{n=1}^{2014} a_{n}
\end{aligned}
$$

(a) 448
(b) 486
(c) 1560
(d) 2001
(e) 2002
35. If on a clock the hour hand and the minute hand are equidistant from the tick mark for the 6 at the bottom of the clock (that is, the angle of each hand), approximately how many seconds past 8:15 is it?
(a) 145
(b) 154
(c) 156
(d) 208
(e) 216

36. Which of the following best describes the values of $x$ for which $x+1 / x \geq 2$ ?
(a) $x>0$
(b) $x>1 / e$
(c) $x>1$
(d) $x>\sqrt{2}$
(e) $x>e$
37. Two different prime numbers between 4 and 18 are chosen. When their sum is subtracted from their product, which of the following numbers could be obtained?
(a) 21
(b) 60
(C) 119
(d) 180
(e) 231
38. If I drive half-way to my destination at $30 \mathrm{~km} /$ hour, how fast (in km/hour) do I have to travel the rest of my journey to make the average speed for the entire journey 60 km/hour?
(a) 90
(b) 97.5
(c) 105
(d) 120
(e) It cannot be done if driving at a finite speed.
39. If $a, b$ and $c$ are positive real numbers such that $a(b+c)=152, b(c+a)=162$ and $c(a+b)=170$, then $a b c=$
(a) 672
(b) 688
(c) 704
(d) 720
(e) 750
40. Let $n$ be a positive integer such that $\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{n}$ is an integer. Which of the following statements is not true:
(a) 2 divides $n$
(b) 3 divides $n$
(c) 6 divides $n$
(d) 7 divides $n$ (e) $n>84$
41. There are 7 black and 3 red marbles. What is the probability that after 5 marbles are drawn, we have drawn all 3 reds?
(a) $1 / 12$
(b) $3 / 7$
(c) $3 / 10$
(d) $3 / 13$
(e) $3 / 5$
42. The graph of the function $f$ is shown at right. How many solutions does the equation $f(f(x))=6$ have?
(a) 2
(b) 4
(c) 5
(d) 6
(e) 7

43. Two circles of radius 1 are next to each other and touching. What is the area of the largest equilateral triangle that can fit within the open space between the two circles?

(a) $\frac{4}{\sqrt{3}}-2$
(b) $\frac{2}{\sqrt{3}}-2$
(c) $\frac{4}{\sqrt{3}}-1$
(d) $\frac{2}{\sqrt{3}}-1$
(e) $\frac{2}{\sqrt{3}}$
44. Suppose you don't know the day of the week, but you are told the following. When the day after tomorrow is yesterday, 'today' will be as far from Sunday as that day was which was 'today' when the day before yesterday was 'tomorrow.'

What is today?
(a) Saturday
(b) Sunday
(c) Monday
(d) Tuesday
(e) Friday
45. A circle centered at $O$ has radius 1 and contains the point $A$. The segment $A B$ is tangent to the circle at $A$ and $\angle A O B=\theta$. If point $C$ lies on $\overline{O A}$ and $\overline{B C}$ bisects $\angle A B O$, then $O C=$
(a) $\sec ^{2} \theta-\tan \theta$
(b) $1 / 2$
(c) $\frac{\cos ^{2} \theta}{1+\sin \theta}$
(d) $\frac{1}{1+\sin \theta}$
(e) $\frac{\sin \theta}{\cos ^{2} \theta}$


## Tie-breaker Question

How many perfect squares (e.g. 1, 4, 9, ...) are divisors of product

$$
1!\cdot 2!\cdot 3!\cdots \cdot 9!\text { ? }
$$

(Write your answer on the backside of the Scantron form below your name, etc. in the green shaded area.)

Answer: 1344

