Do not open this exam until told to do so.

Pepperdine Math Day

November 9, 2013

Exam Instructions and Rules

1. **Write the following information on your Scantron form:**
   Name in NAME box
   Grade in SUBJECT box
   School name in DATE box (and into PERIOD box, if necessary)
   On the back of your Scantron form, write this same information on the first line of the green shaded area.

2. **This exam will last 90 minutes**. It is a **45 question** multiple choice exam. Each question is followed by answers marked A, B, C, D and E. Exactly one answer is correct for each problem. You will use the first 45 spots on the front page of the scantron form to record your answers. **Your answer to the tie-breaker question should be written on the backside of the Scantron form below your name, etc. in the green shaded area.** Your answer to this problem does not count toward your score—it will be used only for tie-breaking.

3. On this exam, **there is no penalty for incorrect answers**, so it is to your advantage to put an answer for each question, especially if you are able to eliminate one or more of the answers as incorrect. Credit will be given only for answers on your scantron form, not for any work written on the exam itself.

4. **Use a number 2 pencil to mark your answer.** Be sure to completely darken each of your penciled-in answers. Extra pencils are available from proctors.

5. There should be enough space between problems (or the backside of pages) to work your solutions. Credit is given only for answers on your Scantron answer form, not for any work written on the exam or scratch paper.

6. Figures are not necessarily drawn to scale.

7. While we certainly don't expect it, any sort of cheating will be dealt with at the discretion of the proctors, and will likely include at least disqualification.

Do not open this exam until told to do so.

1. If Santa Monica were 2 miles south and 14 miles west of Los Angeles, and if Malibu were 1 mile south and 24 miles west of Los Angeles, what would the approximate distance from Malibu to Santa Monica be?

(a) 10 (b) 10.05 (c) 10.10 (d) 10.15 (e) 10.20

2. For how many pairs of real numbers x and y is their arithmetic average $\frac{x+y}{2}$ and their geometric average $\sqrt{xy}$ the same?

(a) None (b) 1 (c) 2 (d) 4 (e) Infinitely many

3. If points A and B are on the parabola $y = 4x^2 + 7x - 1$ and the origin is the point that is exactly halfway between A and B, what is the length of the line segment from A to B?

(a) $2\sqrt{5}$ (b) $5 + \frac{2}{\sqrt{2}}$ (c) $5 + \sqrt{2}$ (d) 7 (e) $5\sqrt{2}$

4. A nine-sided die with faces numbered 1 through 9 is to be rolled twice. What is the probability that the number that comes up on the second roll will be larger than the number that comes up on the first roll?

(a) $\frac{24}{81}$ (b) $\frac{30}{81}$ (c) $\frac{32}{81}$ (d) $\frac{36}{81}$ (e) None of (a) – (d)

5. In the sequence

$$\ldots, a, b, c, d, 0, 1, 1, 2, 3, 5, 8, \ldots$$

each term is the sum of the two terms to its left. Find b.

(a) -3 (b) -1 (c) 0 (d) 1 (e) None of (a) – (d).

6. If $1 < a \leq b$, what is the largest possible value of $\log_a(a/b) + \log_b(b/a)$?

(a) 0 (b) 1 (c) $\sqrt{2}$ (d) 2 (e) 4

7. What is the value of C for which $x^2 + 7x + C = 0$ has exactly one real solution?

(a) $-\frac{7}{2}$ (b) $\frac{7}{2}$ (c) $-\frac{49}{4}$ (d) $\frac{49}{4}$ (e) None of (a) – (d).

8. Let S_n be the sum of the digits of the (base 10) number n in base 7. For example,

$$1000 = \mathbf{2} \cdot 7^3 + \mathbf{6} \cdot 7^2 + \mathbf{2} \cdot 7^1 + \mathbf{6} \cdot 7^0, \text{ that is, } 1000 = (2626)_7,$$

so $S_{1000} = 2 + 6 + 2 + 6 = 16$. What is S_{2013}?

(a) 11 (b) 13 (c) 15 (d) 16 (e) 18

9. If $2^{2x} = 7$, what is 2^{5x+1}?

(a) $50\sqrt{7}$ (b) $32\sqrt{14}$ (c) $98\sqrt{7}$ (d) $49\sqrt{7} + 2$ (e) None of (a) – (d).

10. What is the sum of the three digits in the largest three-digit prime number?

(a) 19 (b) 20 (c) 24 (d) 25 (e) 26

11. A certain ball, when dropped from any height, bounces back up to 2/5 of the original height. If the ball is dropped from 50 feet, bounces back up and continues to bounce up and down, what is the total distance that ball will travel?

(a) $83\frac{1}{3}$ (b) $116\frac{2}{3}$ (c) $156\frac{2}{3}$ (d) $216\frac{2}{3}$ (e) ∞

12. How many integers in $\{1, 2, 3, \ldots, 1000\}$ do not contain any of the digits 1, 2, or 9?

(a) 216 (b) 294 (c) 336 (d) 342 (e) 343

13. Which value is second largest of the following values?

(a) $1^{10} + 2^9 + 3^8 + \cdots + 10^1$ (b) $10!$ (c) 5^{10}

(d) 10^5 (e) 8^8

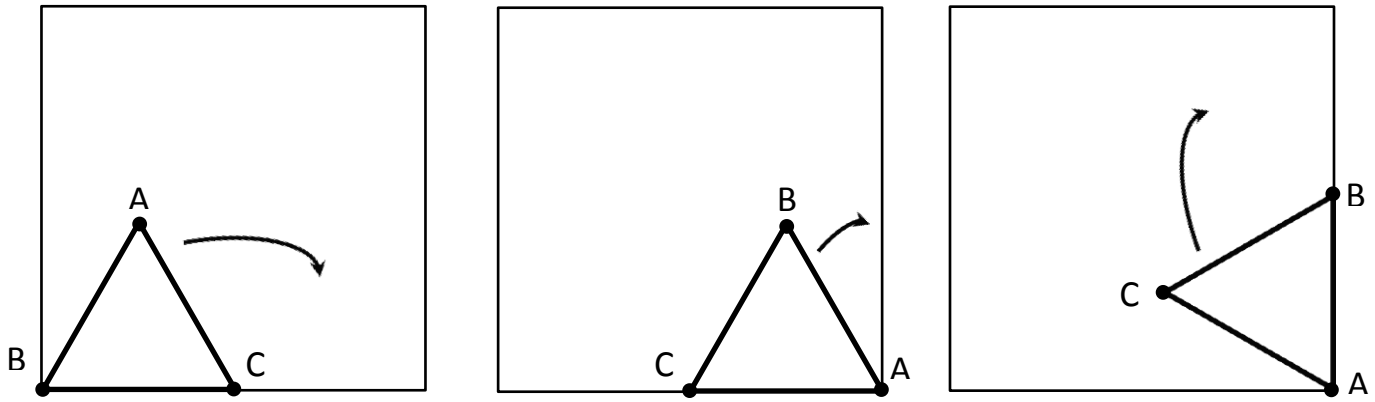
14. Divide the positive integers into groups 1 and 2,3 and 4,5,6 and 7,8,9,10 and so on. What is the sum of the numbers in the 10^{th} group?

(a) 485 (b) 490 (c) 495 (d) 505 (e) 935

15. The expression $(x-1)^4 + 4(x-1)^3 + 6(x-1)^2 + 4(x-1) + 1$ simplifies to:

(a) $(x-2)^4$ (b) x^4 (c) $x^4 - 1$ (d) $(x+1)^4$ (e) $(x-1)^4 - 1$

16. Equilateral triangle ABC with side length 2 inches is placed inside a square with side length 4 inches. The triangle is rotated clockwise about C, and then A, and then B, and so on until A, B, and C are all in their original positions.



The length (in inches) of the path traversed by vertex A is

(a) $10\pi/3$ (b) $20\pi/3$ (c) 32π (d) $40\pi/3$ (e) $80\pi/3$

17. For every triplet $\{a, b, c\}$ of non-zero real numbers we can form the number

$$\frac{a}{|a|}+\frac{b}{|b|}+\frac{c}{|c|}+\frac{abc}{|abc|}.$$

The set of all possible numbers formed is

(a) $\{0\}$ (b) $\{-4, 0, 4\}$ (c) $\{4, -2, 2, 4\}$

(d) $\{-4, -2, 0, 2, 4\}$ (e) None of (a) – (d)

18. In an office, at various times during the day, the boss gives the secretary a letter to type, each time putting the letter on top of the pile in the secretary's in-box. When there is time, the secretary takes the top letter off the pile and types it. If there are five letters in all, and the boss delivers them in the order 1 2 3 4 5, which of the following could <u>not</u> be the order in which the secretary types them?

(a) 1 2 3 4 5 (b) 2 4 3 5 1 (c) 3 2 4 1 5 (d) 4 5 2 3 1 (e) 5 4 3 2 1

19. We have the following statements from Abe, Betty, and Charlie.

Abe: Betty is lying.

Betty: If Charlie is telling the truth, then Abe is lying.

Charlie: Both Abe and Betty are lying.

Who is telling the truth?

(a) Abe only (b) Betty only (c) Betty and Charlie

(d) No one is telling the truth (e) We cannot tell from the given information

20. The letters of the word P E P P E R D I N E can be arranged in $10!/(3!)^2 = 100{,}800$ different ways. If these words are listed alphabetically, what is <u>final</u> letter of the 10^{th} word in the list?

(a) E (b) I (c) N (d) P (e) R

21. Which of the following <u>cannot</u> be the sides of a triangle?

(a) $\sqrt{2}, \sqrt{\sqrt{2}}, \sqrt{\sqrt{\sqrt{2}}}$ (b) $1, e, \pi$ (c) $e, 1, \frac{1}{e}$

(d) $1, \sqrt{2}, \sqrt{3}$ (e) All of (a) – (d) are valid.

22. What fraction of the unit circle $x^2 + y^2 \leq 1$ is contained in the part of the real xy–plane defined by $\frac{x^2}{y^2} \geq 3$?

(a) $\frac{1}{6}$ (b) $\frac{1}{4}$ (c) $\frac{1}{3}$ (d) $\frac{1}{2}$ (e) $\frac{2}{3}$

23. If $x + 3y$ is 10 less than $3x + y$, then $x - y$ is

(a) -10 (b) -5 (c) 5 (d) 10

(e) This cannot be uniquely determined from the given information.

24. If $|a| - |b| = 5$, how many of the following statements could be true (not necessarily simultaneously with the other statements)?

$$a = 0$$

$$b = 0$$

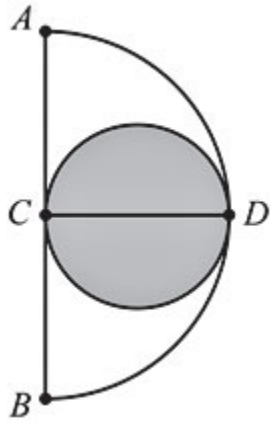
$$a = b$$

$$a = -b$$

$$a = 1$$

(a) None (b) One (c) Two (d) Three (e) Four

25. In the figure at right, a shaded circle with diameter $\overline{CD}$ is tangent to a large semicircle with diameter $\overline{AB}$ at points C and D. Point C is the center of the semicircle and $\overline{AB}$ is perpendicular to $\overline{CD}$. If the area of the large semicircle is 24, what is the area of the shaded circle?



(a) 8 (b) 10 (c) 12

(d) 14 (e) None of (a) – (d).

26. How many roots to $\cos\left(\frac{\pi}{x}\right) = 0$ are there in the region $0.0001 \leq x \leq 1$?

(a) 1001 (b) 9999 (c) 10001 (d) 19999 (e) 20001

27. How many different factors (including 1 and 3960) does 3960 have?

(a) 4 (b) 7 (c) 12 (d) 36 (e) 48

28. A geologist has several rocks of equal weight. If 6 rocks and a 10-ounce weight balance on a scale with 4 rocks and a 22-ounce weight, how many rocks and how much weight would be needed to balance 7 rocks and a 14-ounce weight?

(a) 5 rocks and 26 ounces (b) 9 rocks and 4 ounces (c) 9 rocks

(d) 8 rocks and 10 ounces (e) 6 rocks and 24 ounces

29. Suppose that $f(2n) = 2f(n)$ for all integers n. How many of the following definitions of f could be valid?

$$f(n) = n - 2$$

$$f(n) = n$$

$$f(n) = 2n$$

$$f(n) = 4$$

$$f(n) = 2n - 4$$

(a) One (b) Two (c) Three (d) Four (e) Five

30. A solid box is 15 cm by 10 cm by 8 cm. A new solid is formed by removing a cube 3 cm on a side from each corner of this box. What percent of the original volume has been removed?

(a) 4.5 (b) 9 (c) 13.5 (d) 18 (e) 24

31. If $f(x) = |x - 2| + |2x - 8| + |x - 3|$, what is the largest value of $f(x)$ over $1 \leq x \leq 5$?

(a) 9 (b) 10 (c) 11.5 (d) 12.5 (e) 14

32. $\sqrt{7 + 3\sqrt{5}} - \sqrt{7 - 3\sqrt{5}} =$

(a) $\sqrt{10}$ (b) 10 (c) $\sqrt{3 + 3\sqrt{5}}$ (d) $\sqrt{3 - 3\sqrt{5}}$ (e) $\sqrt{5 + 3\sqrt{5}}$

33. The Catalan number C_n counts the number of ways to write down n pairs of parentheses so that each left parenthesis is followed somewhere by a right parenthesis. For example, $C_2 = 2$ since there are two ways to write down two pairs of parentheses: (()) and ()(). What is C_4?

(a) 2 (b) 5 (c) 10 (d) 12 (e) 14

34. If the graph of the function f in the xy–plane contains the points

$$(0, -9),\ (1, -4),\ (3,\ 0),$$

which of the following cannot be true?

(a) The graph of f has a maximum value.

(b) $y \leq 0$ for all points (x, y) on the graph of f.

(c) The graph of f is symmetric with respect to a line.

(d) The graph of f is a line.

(e) The graph of f is a parabola.

35. Where a and b are real numbers, what it the maximum number of solutions to the system of equations

$$3x + 2y = a$$
$$7x + by = 5$$

depending on what the values of a and b are?

(a) 0 (b) 1 (c) 2 (d) 3 or 4 (e) 5 or more

36. There are two metronomes, sitting side by side. Each metronome is regular, with a fixed amount of time between ticks. One ticks 120 times a minute, the other ticks 25 times a minute. If at some point in time they both tick at precisely the same moment, after how many more seconds do their ticks next coincide?

(a) 5 (b) 10 (c) 12

(d) 24 (e) 60

37. If $(x+y)^2 = x^2 + y^2$, which of the following statements must also be true?

I. $x = 0$

II. $(x-y)^2 = x^2 + y^2$

III. $xy = 0$

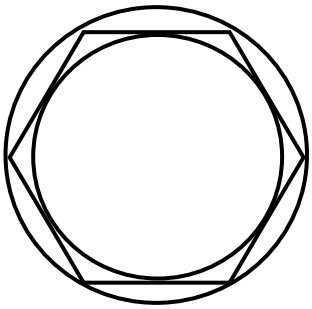
(a) I only (b) II only (c) III only (d) II and III only

(e) None of I, II or III must be true (although one or more statements could be true)

38. All of the students in a class took a 100 point test. Five students scored 100, every student scored at least 60, and the mean (average) score was 76. What is the smallest possible number of students in the class?

(a) 10 (b) 11 (c) 12 (d) 13 (e) 14

39. Inside a circle of radius R, a hexagon of largest possible size is inscribed. Inside of this hexagon, the largest possible circle is then inscribed. What is the area between the two circles?



(a) $\frac{\sqrt{2}}{3}\pi R^2$ (b) $\frac{2\sqrt{2}}{3}\pi R^2$ (c) $\frac{1}{3}\pi R^2$

(d) $\frac{1}{4}\pi R^2$ (e) $\frac{1}{6}\pi R^2$

40. Suppose $x_0 = 2013$, and $x_n = 1 - \frac{1}{x_{n-1}}$ for $n \geq 1$. What is x_{2013}?

(a) $\frac{2012}{2013}$ (b) $\frac{1}{2012}$ (c) $-\frac{1}{2012}$ (d) 2013 (e) -2013

41. How many ordered pairs (x, y) of positive integers are on the line $x + 2y = 100$?

(a) 49 (b) 50 (c) 51 (d) 99 (e) 100

42. If a is the number of purely real solutions to $x^{2013} - 1 = 0$, b is the number of purely imaginary solutions, and c is the number of mixed (part real, part imaginary) solutions, what is $a + b + c$?

(a) 0 (b) 1 (c) 2 (d) 2013 (e) Infinity

43. (This problem is motivated by the real life problem in which one can order 6-piece or 9-piece or 20-piece Chicken McNuggets at McDonalds.) We are interested in numbers that can be written as the sum of 6, 9 and/or 20. For example,

$$35 = 20 + 9 + 6 \quad \text{and} \quad 98 = 20 + 20 + 20 + 20 + 9 + 9 .$$

What is the largest number that cannot be written as the sum of 6, 9 and/or 20?

(a) 43 (b) 57 (c) 59 (d) 63 (e) 71

44. What is the length of the shortest sequence that contains within it all possible sequences of two 0's and two 1's?

(a) 6 (b) 7 (c) 8 (d) 9 (e) 10

45. You are playing a card game with your friend Pete. Pete has brought a non-standard deck of cards. Each of the cards in Pete's deck has a letter on one side and a number on the other. He tells you that the only rule the cards follow is "If there is a vowel on one side, then the other side must have an even number." Pete then deals you the four cards G 4 E 7 (each with the unseen side of the card laying down on the table). Pete is known to occasionally lie, and you are not sure if his rule is true. Which cards must you turn over to determine if Pete is telling the truth about his rule?

(a) E and 4 (b) E and 7 (c) 4 and 7 (d) G and E (e) All of the cards

Tie-breaker Question

Write down your answer on the back of your Scantron form.

How many prime numbers are there between (and including) 2 and 2013?