Do not open this exam until told to do so.

Pepperdine Math Day

November 10, 2012

Exam Instructions and Rules

1. **Write the following information on your Scantron form:**
   Name in NAME box
   Grade in SUBJECT box
   School name in DATE box (and into PERIOD box, if necessary)
   On the back of your Scantron form, write this same information on the first line of the green shaded area.

2. **This exam will last 90 minutes**. It is a **45 question** multiple choice exam. Each question is followed by answers marked A, B, C, D and E. Exactly one answer is correct for each problem. You will use the first 45 spots on the front page of the scantron form to record your answers. **Your answer to the tie-breaker question should be written on the backside of the Scantron form below your name, etc. in the green shaded area.** Your answer to this problem does not count toward your score—it will be used only for tie-breaking.

3. On this exam, **there is no penalty for incorrect answers**, so it is to your advantage to put an answer for each question, especially if you are able to eliminate one or more of the answers as incorrect. Credit will be given only for answers on your scantron form, not for any work written on the exam itself.

4. **Use a number 2 pencil to mark your answer.** Be sure to completely darken each of your penciled-in answers. Extra pencils are available from proctors.

5. There should be enough space between problems (or the backside of pages) to work your solutions. Credit is given only for answers on your Scantron answer form, not for any work written on the exam or scratch paper.

6. Figures are not necessarily drawn to scale.

7. While we certainly don't expect it, any sort of cheating will be dealt with at the discretion of the proctors, and will likely include at least disqualification.

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1. The hypotenuse c and one side a of a right triangle are consecutive integers. The square of the second side is always:

   (a) ca (b) c/a (c) $c+a$ (d) $c-a$ (e) None of (a) – (d).

2. Suppose three trains leave a town at different times and at different speeds (and on different but parallel tracks), as described in the table at right. Which value below is closest to the value of $n \cdot t$, where n is the number of the train which first catches and passes Train 1, and t is the time (in minutes after 1:00) at which that train catches and passes Train 1.

Train	Departure time	Speed
1	1:00	39 mph
2	1:01	59 mph
3	1:02	82 mph

   (a) 4 (b) 6 (c) 8 (d) 9 (e) 10

3. Nine cars containing a total of 30 people passed through a checkpoint. If none of those cars contained more than 4 people, what is the greatest possible number of those cars that could have contained 2 people?

   (a) 1 (b) 2 (c) 3 (d) 4 (e) 5

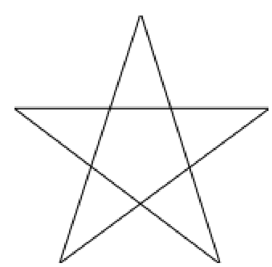
4. If 2012 is written as a product of two positive integers who difference is as small as possible, then the difference of these two integers is:

(a) 2 (b) 49 (c) 51 (d) 104 (e) 499

5. Find the product $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{1}{2012}\right)$.

(a) $1\ \frac{1}{2013}$ (b) $1\ \frac{1}{2012}$ (c) 2 (d) $1006\ \frac{1}{2}$ (e) 2012

6. The least number of marbles one would need in order to place 4 marbles on each line (only one marble per position) of the star figure at right is:



(a) 5 (b) 8 (c) 9

(d) 10 (e) 15

7. Three groups of people will help harvest grapes from a vineyard. Group One can harvest 10 square meters in $1/2$ minute, Group Two can harvest 5 square meters in $1/3$ minute, and Group Three can harvest 20 square meters in 2 minutes. Approximately how many minutes would it take the three groups working together to harvest 200 square meters?

(a) 4 (b) 4.5 (c) 5 (d) 5.5 (e) 6

8. Three cards, each with a positive integer written on it, are lying face-down on a table. Casey, Stacy and Tracy are told that:

(a) The numbers are all different.

(b) Their sum is 13.

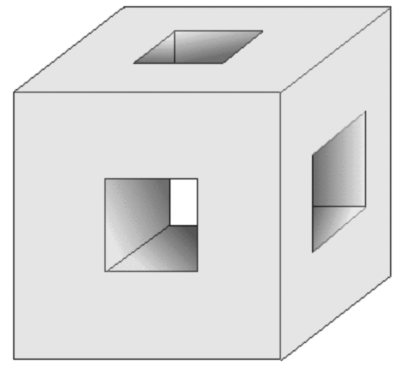
(c) They are in increasing order, from left to right.

First, Casey looks at the number on the leftmost card and says, “I don't have enough information to determine the other two numbers.” Then Tracy looks at the number on the rightmost card and says, “I don't have enough information to determine the other two numbers.” Finally Stacy looks at the number on the middle card and says, “I don't have enough information to determine the other two numbers.” Assume that each person knows that the other two persons can reason perfectly and hears their comments. What number is on the middle card?

(a) 2 (b) 3 (c) 4 (d) 5

(e) There is not enough information to determine the number.

9. In the given figure, a wood cube has edges of length 3. Square holes, each of width 1, centered in each face are cut through to the opposite face. The edges of the holes are parallel to the edges of the cube. Let $S =$ the entire surface area, including the inside, and $V =$ the volume. What is $S - V$?



(a) 31 (b) 32 (c) 36

(d) 51 (e) 52

10. Let $D = a^2 + b^2 + c^2$, where a and b are consecutive integers and $c = ab$. Then $\sqrt{D}$ is:

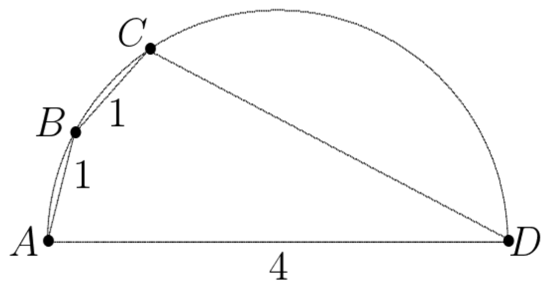
(a) Always an even integer (b) Always an odd integer (c) Always irrational

(d) Sometimes an odd integer, sometimes not (e) Sometimes rational, sometimes not

11. What is the product of the first 2012 terms of a geometric $1, a, a^2, a^3 \ldots$ sequence whose first term is 1 and whose 2012th term a^{2011} is 2?

(a) $\frac{2012 \cdot 2013}{2}$ (b) 2012 (c) 2^{1006} (d) 2^{2011} (e) 2^{2012}

12. Quadrilateral $ABCD$ is inscribed in a circle with side AD, a diameter of length 4. If sides AB and BC each have length 1, then CD has length:



(a) $\frac{7}{2}$ (b) $\frac{5\sqrt{2}}{2}$ (c) $\sqrt{11}$

(d) $\sqrt{13}$ (e) $2\sqrt{3}$

13. Suppose p is an odd integer. Consider the five integers:

p^2

p^3

$p^4 - p^2$

$(p+1)^2 + p^2$

$\frac{p}{2}$ rounded up to the nearest integer

Let m be the number of the above five expressions that must be odd and c be the number that could be odd, but not necessarily, depending on the value of p. (Note: an expression that you count as "must be odd" does not also count as "could be odd"—it is in one category or the other.) What is $2m + c$?

(a) 5 (b) 6 (c) 7 (d) 8 (e) 10

14. If the sum of the three 3-digit numbers below is 789,

$$\begin{array}{r} 1\,2\,3 \\ a\,b\,c \\ +\,d\,e\,f \\ \hline 7\,8\,9 \end{array}$$

and if $1 \leq a, d \leq 9$ and $0 \leq b, c, e, f \leq 9$, what is the smallest possible value of $a + b + c$?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 7

15. Suppose you have a large supply of only pennies, nickels and dimes in a jar. What is the fewest number of coins you would need from the jar to be able to make exact change anywhere from 1 cent to 99 cents?

(a) 10 (b) 11 (c) 12 (d) 13 (e) 14

16. $[(a^{-1} + b^{-1})(a + b)^{-1}]^{-1} =$

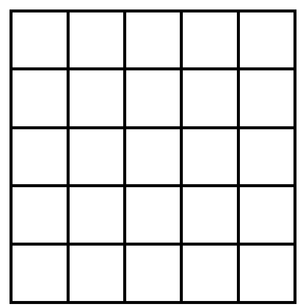
(a) 1 (b) ab (c) $\frac{1}{ab}$ (d) $\frac{b}{a}$ (e) $a^2 + b^2$

17. Consider the increasing infinite sequence of numbers $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$
Is there a smallest real number that exceeds every term of the sequence (that is, a least upper bound)?

(a) No. (b) Yes, and it is 2. (c) Yes, and it is 3.

(d) Yes and it is irrational. (e) Yes, and it is a non-integral rational number.

18. How many different rectangles of any size are there in the figure at right?



(a) 55 (b) 125 (c) 200

(d) 225 (e) 400

19. Which of the following numbers does not divide $2^{1650} - 1$?

(a) 3 (b) 7 (c) 31 (d) 127 (e) 2047

20. What is of the following is closest to the value of the product

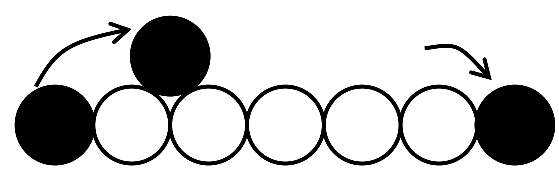
$$(\log_2 4)(\log_3 5)(\log_4 6)(\log_5 7)\cdots(\log_{2009} 2011)(\log_{2010} 2012)?$$

(a) 40 (b) 55 (c) 60 (d) 65 (e) 80

21. The coefficient of x^7 in the polynomial expansion of $(1+2x-x^2)^4$ is:

(a) -12 (b) -8 (c) 6 (d) 12 (e) None of (a) – (d).

22. A circle is rolled, without slipping, across the top of five other identical circles to get from the shown initial position to the shown final position. What is the number of revolutions it must make in the process?



(a) $\frac{3}{\pi}$ (b) $\frac{4}{\pi}$ (c) $\frac{9}{2\pi}$ (d) $\frac{7}{6}$ (e) $\frac{5}{2}$

23. A palindrome is a whole number that reads the same forwards and backwards. If one neglects the colon, certain times displayed on a digital watch are palindromes. Three examples are: 1:01, 4:44, and 12:21. How many times during a 12-hour period will be palindromes?

(a) 56 (b) 57 (c) 60 (d) 63 (e) 90

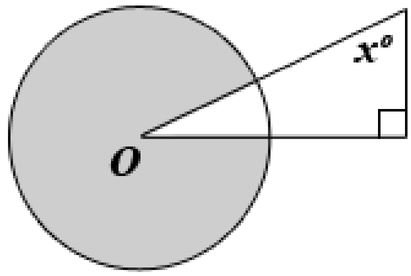
24. The product of the four solutions of the equation $x^4 - 13x^2 - 48 = 0$ is:

(a) -13 (b) 0 (c) 16 (d) 48 (e) None of (a) – (d).

25. A fair die is rolled six times. The probability of rolling at least a 5 at least five times is:

(a) $2/729$ (b) $3/729$ (c) $12/729$ (d) $13/729$ (e) None of (a) – (d).

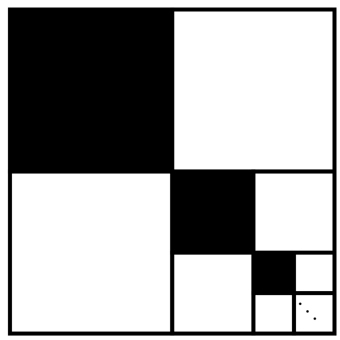
26. The circle shown at right (not drawn to scale) has center O and radius of 5. If the area of the shaded region is 20π, what is the value of x?



(a) 18 (b) 36 (c) 45

(d) 54 (e) 72

27. If the area of the entire box at right is 2, what is the total area of the infinite number of black squares, where the area of each black square is exactly one quarter of the area of the larger square in which it lies, as illustrated in the figure at right? (Note: just the first three black boxes are shown.)



(a) $2/3$ (b) $3/4$ (c) $5/6$

(d) $\sqrt{2}/3$ (e) $\sqrt{2}/4$

28. How many solutions are there to the following system of three equations in three unknowns:

$$x = yz \qquad y = xz \qquad z = xy$$

(a) 2 (b) 4 (c) 5 (d) 6 (e) 8

29. How many different three digit numbers are there in which only the digits 1 – 7 are used and exactly two of the three digits are the same?

(a) 21 (b) 42 (c) 63 (d) 126 (e) 147

30. If a two-digit positive integer is k times the sum of its digits, the number formed by interchanging the digits is the sum of the digits multiplied by:

(a) $9 - k$ (b) $10 - k$ (c) $11 - k$ (d) $k - 1$ (e) $k + 1$

31. What is the 2012th digit after the decimal in the fraction $\frac{1}{7}$?

(a) 1 (b) 2 (c) 4 (d) 5 (e) 7

32. The function f is defined on the set of integers and satisfies

$$f(n) = \begin{cases} n-3 & if\ n \geq 2012 \\ f\big(f(n+5)\big) & if\ n < 2012 \end{cases}$$

Find $f(100)$.

(a) 95 (b) 2009 (c) 2010 (d) 2011 (e) 2012

33. What is the volume of the largest cube that will fit inside of a sphere of radius 2?

(a) $8\sqrt{2}$ (b) $8\sqrt{3}$ (c) $16\sqrt{2}$ (d) $\frac{16\sqrt{2}}{27}$ (e) $\frac{64}{3\sqrt{3}}$

34. If two standard six-sided dice are rolled, what is the probability that the two dice will show different numbers?

(a) 1/6 (b) 10/36 (c) 16/36 (d) 21/36 (e) 5/6

35. A man walks one mile. He then veers 60° to his left and walks another mile. He then veers another 60° to his left and walks another mile. How far is he from where he started?

(a) $\sqrt{2}$ (b) 3/2 (c) 2 (d) $1+\sqrt{3}$ (e) 3

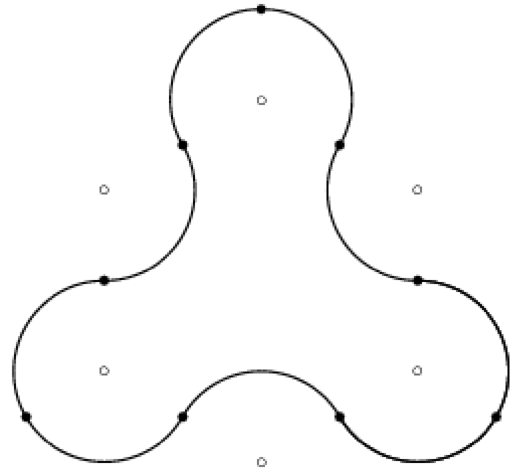
36. The number of different diagonals that can be drawn from one vertex to another inside (not along the edge between adjacent vertices) of a regular polygon of 100 sides is:

(a) 4850 (b) 4950 (c) 9700 (d) 9800 (e) 9900

37. For how many positive two-digit integers is the ones digit <u>greater</u> than <u>twice</u> the tens digit?

(a) 13 (b) 16 (c) 20 (d) 28 (e) 32

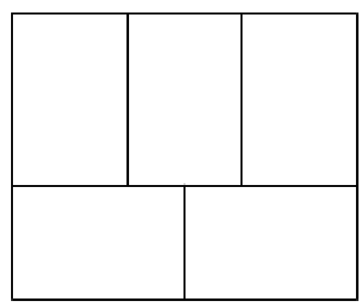
38. The closed curved in the figure is made up of 9 congruent circular arcs each of length $2\pi/3$, where each of the centers of the corresponding circles is among the vertices of a regular hexagon of side 2. What is the area enclosed by the curve?



(a) $2\pi + 6$ (b) $2\pi + 4\sqrt{3}$ (c) $3\pi + 4$

(d) $\pi + 6\sqrt{3}$ (e) $2\pi + 3\sqrt{3} + 2$

39. A rectangle with perimeter 176 is divided into five congruent rectangles as shown in the diagram. What is the perimeter of one of the five congruent rectangles?



(a) 35.2 (b) 76 (c) 80

(d) 84 (e) 86

40. If x inches of wire are used to build the skeleton (i.e the frame) of a cube, what is the surface area of the cube, in square inches?

(a) $\frac{1}{24}x^2$ (b) $\frac{1}{8}x^2$ (c) $\frac{3}{8}x^2$ (d) $2x^2$ (e) $12x^2$

41. What is the area of the shape enclosed by the x-axis, the y-axis and the two lines $y = 5x + 6$ and $y = -x + 20$?

(a) $71\frac{1}{3}$ (b) $164\frac{8}{9}$ (c) $176\frac{2}{3}$ (d) $183\frac{2}{3}$ (e) $329\frac{7}{9}$

42. Five straight lines, none of which is parallel to any other line, are drawn on a plane. If there is not a single point of intersection of all five lines, what is the minimum number of points of intersection?

(a) 4 (b) 5 (c) 10 (d) 15 (e) 20

43. A 24-foot ladder is placed against a vertical wall of a building, with the bottom of the ladder standing on concrete 7 feet from the base of the building. If the top of the ladder slips down 4 feet, then the bottom of the ladder will slide out approximately:

(a) 4 feet (b) 5 feet (c) 6 feet (d) 7 feet (e) 8 feet

44. A 3×3 square is partitioned into 9 unit squares. Each unit square is painted either white or black with each color being equally likely, chosen independently at random. The square is then rotated $90°$ clockwise about its center, and every white square in a position former occupied by a black square is painted black. The colors of all other squares are left unchanged. What is the probability the grid is now entirely black?

(a) $\frac{49}{512}$ (b) $\frac{7}{64}$ (c) $\frac{121}{1024}$ (d) $\frac{81}{512}$ (e) $\frac{9}{32}$

45. What number is three times as large as the square of its reciprocal?

(a) $\sqrt[3]{3}$ (b) $\sqrt{3}$ (c) $\sqrt[3]{2}$ (d) $\sqrt{2}$ (e) $\sqrt[3]{9}$

Tie-breaker Question

Write down on the back of your Scantron form as many digits (in the correct order) of $\sqrt{2}$ as you can.