DO NOT OPEN THIS EXAM UNTIL TOLD TO DO SO.

Pepperdine Math Day

November 14, 2009

Exam Instructions and Rules

1. **Write the following information on your Scantron form:**
   Name in NAME box
   Grade in SUBJECT box
   School name in DATE box (and into PERIOD box, if necessary)
   On the back of your Scantron form, write this same information on the first line of the green shaded area.

2. **This exam will last 90 minutes**. It is a **45 question** multiple choice exam. Each question is followed by answers marked A, B, C, D and E. Exactly one answer is correct for each problem. You will use the first 45 spots on the front page of the scantron form to record your answers. **Your answer to the tie-breaker question should be written on the backside of the Scantron form below your name, etc. in the green shaded area.** Your answer to this problem does not count toward your score—it will be used only for tie-breaking.

3. On this exam, **there is no penalty for incorrect answers**, so it is to your advantage to put an answer for each question, especially if you are able to eliminate one or more of the answers as incorrect. Credit will be given only for answers on your scantron form, not for any work written on the exam itself.

4. **Use a number 2 pencil to mark your answer.** Be sure to completely darken each of your penciled-in answers. Extra pencils are available from proctors.

5. There should be enough space between problems to work your solutions. If needed, extra scratch paper is available at the back of the exam. Credit is given only for answers on your Scantron answer form, not for any work written on the exam or scratch paper.

6. Figures are not necessarily drawn to scale.

7. While we certainly don't expect it, any sort of cheating will be dealt with at the discretion of the proctors, and will likely include at least disqualification.

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1. Halloween (October 31) was on Saturday in 2009. On what day will Halloween be in 2099?

   (a) Saturday   (b) Sunday   (c) Monday   (d) Tuesday   (e) None of (a) – (d)

2. Let f be a function satisfying $f(xy) = \frac{f(x)}{y}$ for all positive real numbers x and y. If $f(500) = 3$, what is the value of $f(600)$?

   (a) 1   (b) 2   (c) $\frac{5}{2}$ (d) 3   (e) $\frac{18}{5}$

3. What is the 9$^{\text{th}}$ digit after the decimal in the fraction $\frac{1}{2009}$?

   (a) 4   (b) 6   (c) 7   (d) 9   (e) None of (a) – (d)

4. You can buy four chocolate bars and three peanut butter bars for $.50 and three chocolate bars and four peanut butter bars for $.48. What is the most candy (greatest number of bars) you can buy for $1.00?

   (a) 14   (b) 15   (c) 16   (d) 17   (e) 18

5. How many squares of any size are there in the figure below? The figure is 10 small squares wide by 10 small squares high.
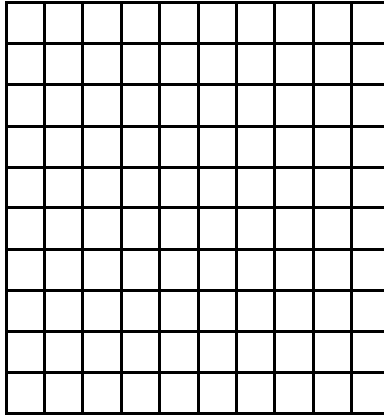
(a) 100   (b) 200   (c) 221   (d) 225   (e) 385

6. Two teams play in the World Series, in which the first team to win four games is the champion (so the series ends after one team has won four games). How many different series can be played?

(a) 35   (b) 56   (c) 70   (d) 112   (e) 114

7. How many different values could $\sin x$ have for which $\sin 10x$ would equal 0?

(a) 10   (b) 11   (c) 19   (d) 20   (e) 21

8. Where $i = \sqrt{-1}$, what is the decimal equivalent of $(4/5)^{i^{434}}$?

(a) 0.5   (b) 0.75   (c) 0.8   (d) 1.0   (e) 1.25

9. To what value does the sequence $1, \sqrt{2}, \sqrt[3]{3}, \ldots, \sqrt[n]{n}, \ldots$ converge?

(a) 0   (b) $\frac{1}{\sqrt{2}}$ (c) 1   (d) $\sqrt{2}$ (e) $\sqrt{3}$

10. If x, y, a and b are real numbers that satisfy $x^a y^b = \left(\frac{3}{4}\right)^{a-b}$ and $x^b y^a = \left(\frac{3}{4}\right)^{b-a}$, what is the product xy?

(a) $\frac{3}{4}$ (b) $-\frac{3}{4}$ (c) 1   (d) -1

(e) More than one of (a) – (d) is possible, depending on the values of a and b.

11. What is the product of the first 10 terms of a geometric $1, a, a^2, \ldots$ series whose first term is 1 and whose tenth term a^9 is 2?

(a) 16   (b) $16\sqrt[9]{2}$ (c) 32   (d) $32\sqrt[9]{2}$ (e) $32\sqrt{2}$

12. In triangle ABC, what is the ordered pair of real numbers (x, y) for which $\sin A : \sin B : \sin C = 4{:}5{:}6$ and $\cos A : \cos B : \cos C = x : y : 2$?

(a) (4,5)   (b) (10,15)   (c) (9,12)   (d) (12,9)   (e) (5,6)

13. The lengths of the sides of a triangle are 3, 4 and 6. What is the least possible perimeter of a *similar* triangle, one of whose sides has length 12?

(a) 15   (b) 21   (c) 26   (d) 39   (e) 52

14. If x and y are *real* numbers such that $x + xi + y - yi = -4$, what is the product xy?

(a) -20 (b) -12 (c) -6 (d) 4   (e) 10

15. A lattice point in the plane is a point with integer coordinates. If the two endpoints are included, how many lattice points lie on the straight line segment connecting (0,0), and (100, 150)?

(a) 50   (b) 51   (c) 66   (d) 75   (e) 76

16. In the following figure (not necessarily drawn exactly to scale),
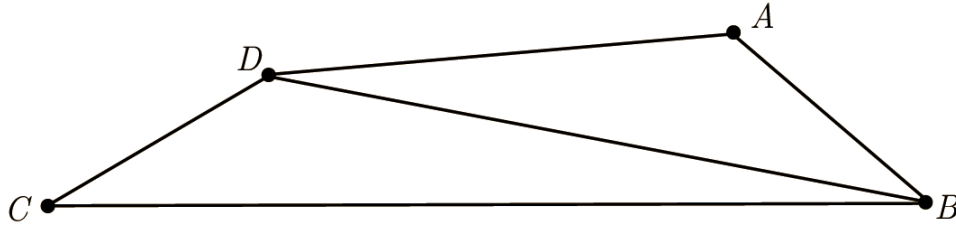
suppose AB = 5, BC = 17, CD = 5, DA = 9 and BD is an integer. What is BD?

(a) 11   (b) 12   (c) 13   (d) 14   (e) 15

17. For every two widgets I buy at regular price, I get a third one for two cents. I spent $.90 for 9 widgets. In cents, what is the regular price of a widget?

(a) 10   (b) 12   (c) 14   (d) 15   (e) 16

18. Let F_n be the nth Fibonacci number where $F_0 = 0$ and $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$. Which of the following is equal to $(F_{n+1})^2 + (F_n)^2$?

(a) $(F_{n+2})^2$ (b) $F_{n^2+(n+1)^2}$ (c) F_{2n} (d) F_{2n+1} (e) F_{2n^2+1}

19. Suppose you have 100 lockers labeled 1 through 100. The lockers are initially all closed.

- The first time you walk by, you open all the lockers
- The second time you walk by you close every other locker (i.e. lockers 2, 4, 6, etc.).
- The third time you walk by you change the status (i.e., you open a locker if it is closed, and close it if it is open) of every third locker (i.e. lockers 3, 6, 9, etc.).
- The next time you change the status of every fourth locker
- The next time every fifth locker, and so on.

If you continue to do this 100 times, which lockers are open at the end?

(a) Only prime numbered lockers   (b) Only non-prime numbered lockers

(b) Only odd numbered lockers   (d) Only even numbered lockers

(e) None of (a) – (d) accurately describes which lockers are open

20. If the first five triangular numbers are 1, 3, 6, 10, 15 and the first five square numbers are 1, 4, 9, 16, 25, what are the first four pentagonal numbers?

(a) 1, 5, 12, 22, 35   (b) 1, 3, 7, 13, 21   (c) 1, 4, 10, 19, 31

(d) 1, 5, 10, 15, 20   (e) 1, 5, 20, 45, 90

21. How many different 8-digit binary sequences are there with six 1's and two 0's?

(a) 56   (b) 8!   (c) 28   (d) 6! 2!   (e) 2^8

22. The circles $x^2 - 2x + y^2 = 9$ and $x^2 + 6x + y^2 = 1$ intersect at two points (x_1, y_1) and (x_2, y_2). What is the sum $x_1 + y_1 + x_2 + y_2$?

(a) -2 (b) -1 (c) $-1 + \sqrt{6}$ (d) $-2 + 2\sqrt{6}$

(e) None of (a) – (d)

23. Two similar square pyramids are built side-by-side. The base of the first pyramid has an area of 40 m^2 and the base of the second pyramid has as area of 120 m^2. If the volume of the first pyramid is 150 m^3, what is the volume of the second pyramid?

(a) $450\sqrt{3}\ m^3$ (b) $450\ m^3$ (c) $1350\ m^3$ (d) $150\sqrt{3}\ m^3$

(e) None of (a) – (d)

24. A triangle is formed with vertices at the points (0,0), (1,2), and (3,1). This triangle can be classified as:

(a) A right triangle   (b) An equilateral triangle

(c) An isosceles (non-equilateral) triangle   (d) Both (a) and (b)

(e) Both (a) and (c)

25. Let S be the set of all points equidistant from the points $(-1, 2)$ and $(5, -3)$. The points in set S form a(n)

   (a) Point   (b) Line   (c) Circle   (d) Ellipse   (e) Hyperbola

26. For any two integers, a and b, we will define a relationship $\sim$ by saying that $a \sim b$ if $a - b$ is divisible by 3. Which of the following relationships is true for every integer x?

   (a) $x \sim 2x$ (b) $x \sim 3x$ (c) $x \sim 4x$ (d) $x \sim x^3$ (e) $x + 1 \sim x + 2$

27. 30% of P is 5% of 360. What is P?

   (a) 5.4   (b) 54   (c) 60   (d) 108   (e) 600

28. On the standard xy-plane, what is the area bounded by $|3x - 18| + |2y + 7| < 3$?

   (a) 3   (b) $\frac{7}{2}$ (c) 4   (d) $\frac{9}{4}$ (e) 5

29. Suppose $f(x) = x^2 + px + q$ is a quadratic function such that the numbers p and q satisfy $p^2 = 4q$. How many roots are there to $f(x) = 0$?

   (a) One real double root

   (b) Two complex (imaginary) roots

   (c) Two real roots

   (d) One real and one imaginary root

   (e) We can't tell from the information given.

30. If the five given expressions in (a) – (e) were list from smallest to largest, which would be the middle value?

   (a) $\log_{10}(2009^{2009})$

   (b) 2009^{10}

   (c) $(200 + 9)!$

   (d) 10^{2009}

   (e) $\sqrt[2009]{2009!}$

31. If the hypotenuse of a right triangle has length 2009, and x and y are the lengths of the other two sides, what is the maximum number of the following five statements that *could* be *simultaneously* true?
   - $x + y \geq 2009$
   - $xy \leq 2009$
   - $x^y \leq 2009$
   - $y^x \geq 2009$
   - $|x - y| \leq 2009$

   (a) 1   (b) 2   (c) 3   (d) 4   (e) 5
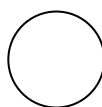
32. How many *real* solutions are there to $x^{2009} - \frac{1}{x^{2009}} = 1$?

    (a) 0   (b) 1   (c) 2   (d) 3   (e) 4 or more

33. The top and front views of a 3-dimensional object are given below.
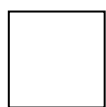
      Top View   Front and Back View

Of the choices in (a) – (e), this object would best be described as a

    (a) Cylinder   (b) Sphere   (c) Cube   (d) Oval   (e) Cone

34. Of the five given numbers, which is closest to the value of the expression

$$\sqrt{2009 + \sqrt{2009 + \sqrt{2009 + \cdots}}}$$

    (a) 40   (b) 45   (c) 50   (d) 55   (e) 60

35. Suppose that of 2009 balls in a bin, 9 are blue and 2000 are orange. Suppose you select one ball, then another, then a third, *without looking* at any of the three. What is the probability that third ball is blue?

(a) $\dfrac{7}{2007}$ (b) $\dfrac{8}{2007}$ (c) $\dfrac{7}{2007}$ (d) $\dfrac{8}{2008}$ (e) $\dfrac{9}{2009}$

36. Consider the maze, shown at right, which is 10 units by 10 units square. What is the length of the path from the *center* of the first square inside the maze to the *center* of the most interior square, if the path is always halfway between the "walls" of the maze?
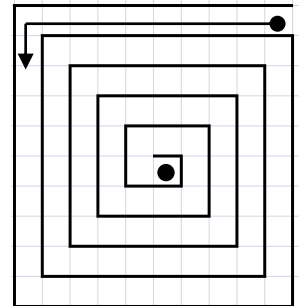
(a) 99   (b) 99.5   (c) 100   (d) 110   (e) 120

37. If you have 50 coins worth $1.00, and you randomly select one coin and find that it is a quarter, which you put back into the collection of coins. You then select another coin. What is the probability that this next coin is a penny?

(a) .45   (b) .80   (c) .85   (d) .90   (e) None of (a) – (d)

38. If the sum of the three 3-digit numbers below is 789,

$$\begin{array}{r} 1\,2\,3 \\ a\,b\,c \\ +\,d\,e\,f \\ \hline 7\,8\,9 \end{array}$$

and if $1 \le a, d \le 9$ and $0 \le b, c, e, f \le 9$, what is the maximum possible value of $a + b + c$?

(a) 18   (b) 19   (c) 22   (d) 23   (e) 27

39. Suppose that Shaq makes 50% of his free throws while Kobe makes 80% of his free throws. Kobe, needing an ego boost, challenges Shaq to a free throw shooting contest in which the first person to make a free throw wins. Kobe even allows Shaq to shoot first, and then they alternate who shots after that (Shaq, Kobe, Shaq, Kobe, etc). What is the probability that Shaq is the first one to make a free throw?

(a) $\frac{1}{2}$ (b) $\frac{5}{9}$ (c) $\frac{3}{5}$ (d) $\frac{4}{7}$ (e) $\frac{10}{11}$

40. What are the last two digits of 5^{2009}?

(a) 05   (b) 25   (c) 55   (d) 65   (e) 75

41. Define a sequence by $a_1 = 2$ and $a_{n+1} = \frac{1+a_n}{1-a_n}$ for $n > 1$. What is a_{2009}?

(a) -2009 (b) -3 (c) $-\frac{1}{2}$ (d) 2 (e) 3

42. What is the largest power of 2 that divides $2^{2009} + 10^{2009}$?

(a) 2^{2009} (b) 2^{2010} (c) 2^{2011} (d) 2^{2012} (e) 2^{2013}

43. Suppose you drop a ball from a height of one meter and the ball bounces back to a height of three quarters of a meter. Each time the ball bounces back up to three quarter the previous height. If you let the ball continue to until it comes to rest, which value below is closest to how far will the ball have traveled?

(a) $4\,m$ (b) $5\,m$ (c) $6\,m$ (d) $7\,m$ (e) $8\,m$

44. Suppose a circle is divided into 2009 wedges (like slicing a pie or cake) with equal angles. If $x = 0.abc\ldots$ is the measure of each of those angles, measured in *degrees* (and note that $0 < x < 1$) and a, b, c are the first three digits after the decimal, what is the sum $a + b + c$?

    (a) 3   (b) 5   (c) 13   (d) 15   (e) 17

45. Suppose a sphere with diameter of 10 units is sliced into 10 parallel slices each of exactly 1 unit width (to form 10 flat, circular objects, like 10 pizzas, of varying diameters). What is the total surface area of all of the slices?

    (a) 160π (b) 210π (c) 270π (d) 330π (e) 430π

Tie-breaker Question

**Write down on the back of your Scantron form
as many digits (in the correct order) of π as you can.**