DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO.

Pepperdine Math Day 2006

October 28, 2006

Exam Instructions and Rules

1. **Write the following information on your Scantron:**
   Name in NAME box
   Grade in SUBJECT box
   Exam color in TEST NO. box
   School name in DATE box (into PERIOD box, if necessary)

   On the back of the Answer Form, put this same information in the same order.

2. **This exam will last approximately 90 minutes**. It is a **50 question** multiple choice exam. Each question is followed by answers marked A, B, C, D and E. Exactly one answer is correct for each problem. You will use the first 50 spots (the entire front page only) on the scantron form to record your answers.

3. On this exam, **there is no penalty for incorrect answers**, so it is to your advantage to put an answer for each question, especially if you are able to eliminate one or more of the answers as incorrect. However, **in the case of a tie, the person with fewer incorrect answers wins**. If there is still a tie, your answers to problem 50 will determine who wins, then problem 49, and so on, if necessary. Credit will be given only for answers on your scantron form, not for any work written on the exam itself.

4. **Use a number 2 pencil to record your answer.** Be sure to completely darken each of your penciled-in answers. Extra pencils are available from proctors.

5. There should be enough space between problems to work your solutions. If needed, extra scratch paper is available from the proctors. Of course, credit is given only for answers on your answer form, not for any work written on the exam or scratch paper.

6. Figures are not necessarily drawn to scale.

7. While we certainly don't expect it, any sort of cheating will be dealt with at the discretion of the proctors, and will likely include at least disqualification.

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1. Two trains 150 miles apart are traveling toward each other along the same track. The first train is traveling at 30 miles per hour, and the second train is traveling at 45 miles an hour. A fly is hovering just above the nose of the first train. It flies from the first train to the second train, turns around immediately, flies back to the first train, then turns around again. It goes on flying back and forth between the two trains until they are only 50 miles apart. If the fly's speed is 120 miles per hour, how far will it travel by the time the trains are 50 miles apart?

   (a) 100 (b) 160 (c) 180 (d) 240 (e) ∞

2. What is the sum of the digits in the number $10^{2006} - 2006$ when expressed as a single whole number?

   (a) 18038 (b) 18047 (c) 18054 (d) 18055 (e) 18056

3. How many times will a common clock (that strikes only on the hour, e.g. at 3:00 it strikes 3 times) strike in a week?

   (a) 462 (b) 546 (c) 924 (d) 1092 (e) 1104

4. What is the value of $\sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}$?

   (a) $\sqrt{8}$ (b) 3 (c) $\sqrt{12}$ (d) 4 (e) ∞ (not convergent)

5. Suppose two baskets contain some red and white balls:

Basket A contains 2 red and 3 white balls.

Basket B contains 3 red and 4 white balls.

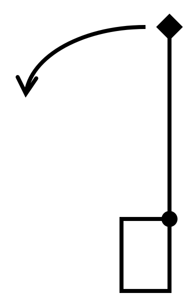
Suppose 1 ball is randomly drawn from basket A and 2 are drawn from basket B. How likely is it that at least one of the balls drawn is white?

(a) $\frac{12}{35}$ (b) $\frac{24}{35}$ (c) $\frac{25}{35}$ (d) $\frac{33}{35}$ (e) $\frac{34}{35}$

6. Suppose $x_0 = 1776$, and $x_n = 1 - \frac{1}{x_{n-1}}$ for $n \geq 1$. What is x_{2006}?

(a) $\frac{1775}{1776}$ (b) $\frac{1}{1775}$ (c) $-\frac{1}{1775}$ (d) 1776 (e) -1776

7. A dog is tied to the corner of a 10-foot wide by 15-foot long shed on a rope of length 50 feet. Assume the dog starts out as pictured and winds his way around the shed counter-clockwise as far as he can go, always keeping the rope as tight as possible. What is the total area (in square feet) swept out by the rope?



(a) 625π (b) 1087.5π (c) 1237.5π

(d) 2500π (e) ∞ (the dog goes around forever)

8. The average of 8 different positive whole numbers is 8. What is the largest possible value of any of these numbers?

(a) 36 (b) 55 (c) 56 (d) 57 (e) 64

9. Where $f(x) = ax^{13} + bx^9 - cx^5 + 3$, if $f(-5) = 5$, what is the value of $f(5)$?

(a) -5 (b) 1 (c) 5 (d) 15

(e) We cannot tell from the information given.

10. The set of all real numbers x for which

$$\log_{2006}(\log_{2005}(\log_{2004} x)))$$

is defined is $\{x : x > c\}$. What is the value of c?

(a) 0 (b) 1 (c) 2004 (d) 2004^{2005} (e) $2004^{2005^{2006}}$

11. When 3^{2006} is divided by 5, what is the remainder?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

12. Find the smaller angle formed by the intersection of the two lines $y = \sqrt{3}x - 5$ and $\sqrt{2}x + \sqrt{2}y = \sqrt{7}$.

(a) 45^{o} (b) 52.5^{o} (c) 60^{o} (d) 67.5^{o} (e) 75^{o}

13. What is the maximal number of points of intersection of a fifth degree polynomial and a seventh degree polynomial?

(a) 2 (b) 5 (c) 7 (d) 12 (e) 35

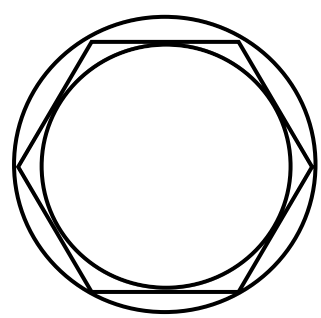
14. Which of the following are real solutions to $x^4 - 2x^2 - 1 = 0$?

(a) $1 \pm \sqrt{2}$ (b) $+\sqrt{1 \pm \sqrt{2}}$ (c) $\pm\sqrt{1+\sqrt{2}}$ (d) $\pm\sqrt{1-\sqrt{2}}$ (e) $\pm\sqrt{1 \pm \sqrt{2}}$

15. If $\boldsymbol{f}(\boldsymbol{x}) = |\boldsymbol{x} - 2| + |2\boldsymbol{x} - 8| + |\boldsymbol{x} - 3|$, for $1 \le \boldsymbol{x} \le 5$, what is the largest value of $\boldsymbol{f}(\boldsymbol{x})$?

(a) 1 (b) 3 (c) 5 (d) 7 (e) None of (a) – (d)

16. Inside a circle of radius *R*, a hexagon of largest possible size is inscribed. Inside of this hexagon, the largest possible circle is then inscribed. What is the area between the two circles?



(a) $\frac{\sqrt{2}}{3}\pi\boldsymbol{R}^2$ (b) $\frac{\sqrt{3}}{2}\pi\boldsymbol{R}^2$ (c) $\frac{1}{3}\pi\boldsymbol{R}^2$

(d) $\frac{1}{4}\pi\boldsymbol{R}^2$ (e) $\frac{1}{6}\pi\boldsymbol{R}^2$

17. Suppose $g(x) = 1 - x^2$ and $f(g(x)) = \dfrac{x^2}{(1-x^2)^2}$ for $x \neq 1$. Find the value of $f(2)$.

(a) $-\dfrac{1}{4}$ (b) -1 (c) 1 (d) -4 (e) 4

18. Suppose $f(x+y) = f(x) + f(y) + 1$ and $f(1) = 1$. Find the value of $f(10)$.

(a) 9 (b) 10 (c) 11 (d) 19

(e) $f(10)$ is not uniquely determined by the given information.

19. A container has four quarts of maple syrup. One quart is removed and replaced with one quart of corn syrup, and the result is thoroughly mixed. One quart of the current mixture is removed and replaced with corn syrup. This process is repeated two more times. How much maple syrup has been taken out by these four removals?

(a) Between 2 and 2 ¼ quarts (b) Between 2 ¼ and 2 ½ quarts

(c) Between 2 ½ and 2 ¾ quarts (d) Between 2 ¾ and 3 quarts

(e) Between 3 and 3 ¼ quarts

20. Mike runs at a speed of 12 miles per hour. David runs at a speed of 10 miles per hour. They start at the same place on a circular track 1/4 mile long and run in opposite directions. How many miles has David run when they meet for the 20th time?

(a) $\frac{10}{4}$ (b) $\frac{11}{4}$ (c) $\frac{25}{11}$ (d) $\frac{25}{10}$ (e) 5

21. A particular die has its six faces labeled 1, 2, 3, 5, 7 and 9. If two such dice are rolled and the numbers showing face up are added, what is the total number of possible different sums?

(a) 11 (b) 12 (c) 13 (d) 14 (e) 36

22. For what real numbers $\boldsymbol{p}$ is $\boldsymbol{px}^2 + \boldsymbol{px} - 1 \le 0$ for all $\boldsymbol{x}$?

(a) $\boldsymbol{p} \le -4$ (b) $-4 \le \boldsymbol{p} \le 0$ (c) $\boldsymbol{p} \le 0$ (d) $-4 \le p \le 0$ (e) None of (a) – (d)

23. Find the sum $\boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{S}_3 + \cdots + \boldsymbol{S}_{10}$ where $\boldsymbol{S_k} = (\boldsymbol{k} + 1)^2 - (\boldsymbol{k} - 1)^2$.

(a) 120 (b) 121 (c) 216 (d) 220 (e) 221

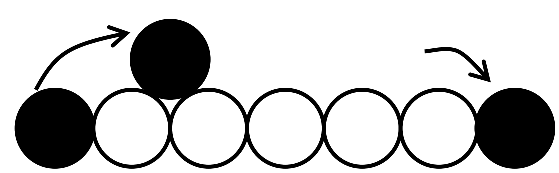
24. If $a = \dfrac{1}{2-\sqrt{3}}$ and $b = \dfrac{1}{2+\sqrt{3}}$, find the value of $7a^2 + 11ab - 7b^2$.

(a) $11 + 56\sqrt{3}$ (b) $11 + 14\sqrt{3}$ (c) $18 + 14\sqrt{3}$ (d) $18 - 14\sqrt{3}$ (e) $18 + 56\sqrt{3}$

25. A man walks one mile. He then veers 30° to his left and walks another mile. He then veers another 30° to his left and walks another mile. How far is he from where he started?

(a) $1 + \sqrt{3}$ (b) $2 + \sqrt{3}$ (c) $\sqrt{1+\sqrt{3}}$ (d) $\sqrt{2+\sqrt{3}}$ (e) $\sqrt{4+2\sqrt{3}}$

26. A circle is rolled, without slipping, across the top of five other identical circles to get from the shown initial position to the shown final position. What is the number of revolutions it must make in the process?



(a) $\dfrac{7}{6}$ (b) $\dfrac{8}{6}$ (c) $\dfrac{16}{6}$ (d) $\dfrac{19}{6}$ (e) $\dfrac{20}{6}$

27. The first five pentagonal numbers, as illustrated at right, are 1, 5, 12, 22, and 35. The tenth pentagonal number is:

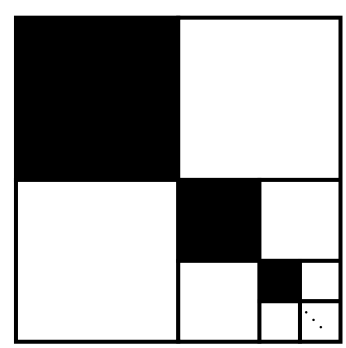
(a) 115 (b) 117 (c) 125 (d) 145 (e) 176

28. A test consists of ten questions. 10 points are given for each correct answer, 3 points are deducted for each incorrect answer, and no points are deducted for unanswered questions. If John has a score of 61, how many questions did he answer correctly?

(a) 6 (b) 7 (c) 8 (d) 9

(e) We cannot tell from the information given.

29. If the area of the entire box at right is 1, what is the total area of the infinite number of black squares, where the area of each black square is exactly one quarter of the area of the larger square in which it lies, as illustrated in the figure at right? (Note: just the first three black boxes are shown.)



(a) $\frac{1}{3}$ (b) $\frac{9}{24}$ (c) $\frac{5}{12}$

(d) $\frac{\sqrt{2}}{6}$ (e) $\frac{\sqrt{2}}{4}$

30. In your room, you have a large supply of pennies, nickels, dimes and quarters in a jar. What is the fewest number of coins you would need from the jar to be able to make exact change anywhere from 1 cent to 99 cents?

(a) 9 (b) 10 (c) 11 (d) 12 (e) 13

31. How many three-digit numbers (i.e. between 100 and 999) are divisible by 3, 5 or 7?

(a) 9 (b) 480 (c) 488 (d) 600 (e) 608

32. A tank is supplied with water by three pipes. The smallest pipe can fill the tank in 40 minutes, the midsized pipe in 30 minutes, and the largest pipe in 20 minutes. To the nearest minute, how many minutes will it take to fill the tank if all three pipes are running at once?

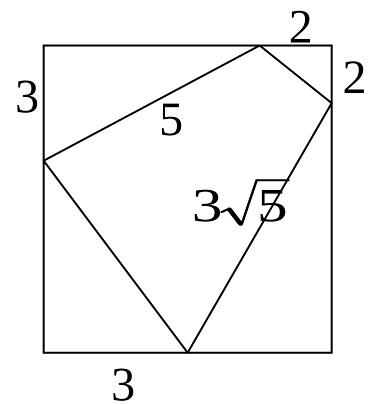
(a) 8 (b) 9 (c) 10 (d) 11 (e) 12

33. The surface area of a cube would increase by 44% if each side length were increased by what percentage amount?

(a) 20% (b) 22% (c) $\sqrt{44}\,\%$ (d) $\sqrt[3]{44}\,\%$

(e) It would depend on the original size of the cube.

34. In the rectangle at right, what is the area of the quadrilateral?



(a) 23.5 (b) 24 (c) 24.5

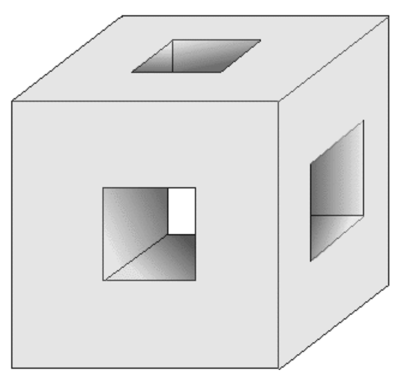
(d) 25 (e) 26.5

35. What point (x, y) on the line $-3x+2y=4$ is nearest the circle $(x-3)^2+(y+1)^2=4$?

(a) $(\frac{1}{3}, \frac{5}{2})$ (b) $(-\frac{1}{3}, \frac{3}{2})$ (c) $(-\frac{6}{13}, \frac{17}{13})$ (d) $(0, 2)$

(e) More than one point—the line intersects the circle.

36. In the given figure, a wood cube has edges of length 3. Square holes, each of width 1, centered in each face are cut through to the opposite face. The edges of the holes are parallel to the edges of the cube. The entire surface area, including the inside, is:



(a) 48 (b) 72 (c) 73

(d) 76 (e) 77

37. Find the product $(1+\frac{1}{5})(1+\frac{1}{6})(1+\frac{1}{7})(1+\frac{1}{8})(1+\frac{1}{9})$.

(a) $\frac{30000}{15120}$ (b) $\frac{30120}{15120}$ (c) 2 (d) $\frac{30360}{15120}$ (e) 3

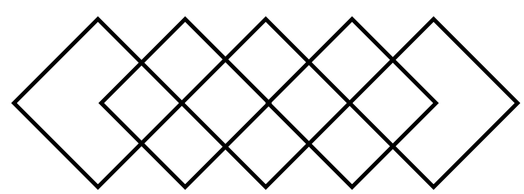
38. The yearly change in the yield of a particular crop over the past 4 years was: 25% increase the first year, followed by a 25% decreased the second year, followed by another 25% decrease the third year, followed by a 25% increase the fourth year. What is the total percentage change (whether positive or negative), to the nearest 1%, in yield at the end of the fourth year compared to the original yield?

(a) 0 (b) 11 (c) 12 (d) 13 (e) 14

39. What is the area of the region in the Cartesian plane described by $|2\boldsymbol{x}|+|3\boldsymbol{y}|\le 6$?

(a) 3 (b) 6 (c) 12 (d) 24 (e) ∞

40. How many rectangles of any size are there in the figure at right?



(a) 28 (b) 29 (c) 30

(d) 31 (e) 32

41. In the series

$$1+1-2+1-2+3+1-2+3-4+1-2+3-4+5+1-2+3-4+5-6+1-\cdots,$$

what is the sum of the first 190 terms?

(a) 0 (b) 10 (c) 95 (d) 190 (e) 380

42. Where a and b are real numbers, what it the maximum number of solutions the system of equations at right might have, depending on what the values of a and b are?

$$3x+2y=a$$
$$7x+by=5$$

(a) 0 (b) 1 (c) 2 (d) 3 or 4 (e) 5 or more

43. If $\boldsymbol{x}=\boldsymbol{t}^{1/(\boldsymbol{t}-1)}$ and $\boldsymbol{y}=\boldsymbol{t}^{\boldsymbol{t}/(\boldsymbol{t}-1)}$, $t>0,\ t\neq 1$, which of the following is a valid relationship between x and y?

(a) $\boldsymbol{y}^{\boldsymbol{x}}=\boldsymbol{x}^{1/\boldsymbol{y}}$ (b) $\boldsymbol{y}^{1/\boldsymbol{x}}=\boldsymbol{x}^{\boldsymbol{y}}$ (c) $\boldsymbol{x}^{\boldsymbol{y}}=\boldsymbol{y}^{\boldsymbol{x}}$ (d) $\boldsymbol{x}^{\boldsymbol{x}}=\boldsymbol{y}^{\boldsymbol{y}}$ (e) None of (a) – (d)

44. If a, b, c and d are positive real numbers such that $1\le\frac{a}{b}<\frac{c}{d}$, which of the following statements is necessarily FALSE.

(a) Either $\boldsymbol{a}<\boldsymbol{c}$ or $\boldsymbol{b}>\boldsymbol{d}$ (b) $\boldsymbol{bc}>\boldsymbol{da}$ (c) $\boldsymbol{b}<\boldsymbol{a}$

(d) $\frac{\boldsymbol{a}}{\boldsymbol{b}}<\frac{\boldsymbol{a}+\boldsymbol{c}}{\boldsymbol{b}+\boldsymbol{d}}$ (e) $\boldsymbol{a}+\boldsymbol{c}<\boldsymbol{b}+\boldsymbol{d}$

45. Cannon balls are stacked in the form of a pyramid with a square base. Suppose that the edges of the square base consist of eight cannon balls (so a total of 64 balls on the bottom layer of the pyramid). Find the total number of balls in the pyramid.

(a) 116 (b) 120 (c) 124 (d) 200 (e) 204

46. In the previous problem, if the radius of each ball is 1, what is the total height of the pyramid?

(a) $8\sqrt{2}$ (b) $8\sqrt{3}$ (c) $2+8\sqrt{3}$ (d) $2+7\sqrt{2}$ (e) $2+7\sqrt{3}$

47. Suppose that $\tan \boldsymbol{x} + \sec \boldsymbol{x} = 2$ for some real number *x*. Then $\tan \boldsymbol{x} - \sec \boldsymbol{x}$ could equal what value?

(a) 0 (b) $-\frac{1}{2}$ (c) $\frac{1}{2}$ (d) -2 (e) 2

48. If all integers from 1 to 1,000,000 are printed, how many times will the digit 7 be written? (For example, in 747,237, the digit 7 appears three times.)

(a) 599,930 (b) 599,937 (c) 600,000 (d) 999,930 (e) 999,937

49. Which of the following have exactly the same graph?

I. $\boldsymbol{y} = \boldsymbol{x} - 2$ II. $\boldsymbol{y} = \dfrac{\boldsymbol{x}^2 - 4}{\boldsymbol{x} + 2}$ III. $(\boldsymbol{x} + 2)\boldsymbol{y} = \boldsymbol{x}^2 - 4$

(a) I and II only (b) I and III only (c) II and III only

(d) I, II and III (e) None of I, II or III—all have different graphs.

50. Jimmy comes home and the power is out. He needs to open the front door. He has three keys, but he can't tell in the dark which key is which. One key opens the door after trying it for 10 seconds. Another key will not open the door, but it takes 5 seconds to discover that. The third key also will not open the door, but it takes 8 seconds to discover that about this key. Jimmy tries one key at random. Perhaps it works (after 10 seconds). If he discovers it doesn't work (after either 5 or 8 seconds), he puts it back, jumbles the keys and tries again with a random key, perhaps the same one he just tried. He continues in this way until the door is open. On average (i.e. if this scenario were to occur several times), how many seconds does it take for Jimmy to open the door?

(a) 16.5 (b) 19.25 (c) 23 (d) 29.5 (e) Forever