# Learning determinants from Cramer \& Cauchy: a TRIUMPHS Primary Source Project 

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# TRIUMPHS 

## TRansforming Instruction in Undergraduate Mathematics via Primary Historical Sources (NSF grant no. 1524065)

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## TRIUMPHS' Goal

To motivate, provide context for, and engage in the study of mathematics by undergraduates at all levels through close reading of primary source materials and carefully designed student tasks based on these texts.

## Goal of this Project

To present a meaningful definition of, and the fundamental properties of the matrix determinant, a difficult concept to teach in a unified manner because of its symbolic complexity.

## The Plan

Present the determinant in a three-week lesson, using two important works in the history of the development of the determinant:

* Gabriel Cramer's publication of his eponymous Rule in an appendix of his Introduction à l'Analyse des Lignes Courbes Algébriques [An Introduction to the Analysis of Algebraic Curves] (1750)
* Augustin-Louis Cauchy's monumental memoir Functions qui ne peuvent obtenir que deux valeurs égales et de signes contraires par suit des transpositions opérées entre les variables qu'elles renferment [On functions which take only two values, equal but of opposite sign, by means of transpositions performed among the variables which are contained therein] (1815), read to the Paris Academy in 1812 - at age 23(!)


## Students Reading and Writing

## Focus on student activity

- Students read portions of these primary texts (in English translation)
- ...presented with commentary text that establishes context, highlights important features, and makes connections with traditional modes of description (notation, terminology)
- ... and carefully crafted student tasks to aid in the construction of robust conceptual understanding.


## First Implementation

## The Course

- Colin McKinney taught MATH 223 Elementary Linear Algebra at Wabash College during Fall 2018 (11 students)
- Used Leon's Linear Algebra with Applications (9/e) as main textbook
- Earlier in the semester, employed another TRIUMPHS PSP, Solving a System of Linear Equations Using Ancient Chinese Methods, by Mary Flagg, to introduce solutions of systems of equations.
- Current PSP introduced in week 5 of the semester. Below, roman numerals indicate class days of implementation.


## A Rule for Solving Linear Systems

I. Cramer presents formulas for the solutions to a linear system of $n$ equations in $n$ variables for $n=1,2,3$

- Here are his formulas for $n=3$ : if the system is

$$
\begin{aligned}
& A^{1}=Z^{1} z+Y^{1} y+X^{1} x \\
& A^{2}=Z^{2} z+Y^{2} y+X^{2} x \\
& A^{3}=Z^{3} z+Y^{3} y+X^{3} x
\end{aligned}
$$

(variables in lowercase), then the solutions are given by

$$
\begin{aligned}
& Z=\frac{A^{2} Y^{2} X^{3}-A^{\mathrm{I}} Y^{3} X^{2}-A^{2} Y^{\mathrm{I}} X^{3}+A^{2} Y^{3} X^{1}+A^{3} Y^{4} X^{2}-A^{3} Y^{2} X^{\mathrm{r}}}{Z^{2} Y^{2} X^{3}-Z^{1} Y^{3} X^{2}-Z^{2} Y^{1} X^{3}+Z^{2} Y^{3} X^{2}+Z^{1} Y^{1} X^{2}-Z^{3} Y^{2} X^{2}} \\
& Z^{1} A^{2} X^{3}-Z^{1} A^{3} X^{2}-Z^{2} A^{1} X^{3}+Z^{2} A^{3} X^{1}+Z^{3} A^{1} X^{2}-Z^{3} A^{2} X^{5} \\
& y=\frac{Z^{2} Y^{2} X^{3}-Z^{1} Y^{3} X^{2}-Z^{2} Y^{1} X^{3}+Z^{2} Y^{3} X^{\mathrm{x}}+Z^{3} Y^{2} X^{2}-Z^{3} Y^{2} X^{\mathrm{t}}}{} \\
& x=\frac{Z Y^{2} A^{3}-Z^{1} Y^{3} A^{2}-Z^{2} Y^{1} A^{3}+Z^{:} Y^{3} A^{1}+Z^{3} Y^{1} A^{2}-Z^{3} Y^{2} A^{3}}{Z^{1} Y^{2} X^{3}-Z^{1} Y^{3} X^{2}-Z^{2} Y^{1} X^{3}+Z^{2} Y^{3} X^{1}+Z^{3} Y^{1} X^{2}-Z^{Y} Y^{Y} X^{1}}
\end{aligned}
$$

## A Rule for Solving Linear Systems

II. Students solve directly to derive Cramer's Rule for $n=2,3$.

- Cramer describes his solution formulas:
we will find the value of each unknown by forming $n$ fractions of which the common denominator has as many terms as there are diverse arrangements of $n$ different things. Each term is composed of the letters $Z Y X V$ \&c., always written in the same order, but to which we distribute, as exponents, the first $n$ numbers arranged in all possible ways.
- Signs are assigned to each such term according to the parity of the derangement of $1,2, \ldots, n$ given by the sequence of "exponents"...


## A Rule for Solving Linear Systems

The common denominator being thus formed, we will have the value of $z$ by giving to this denominator the numerator which is formed by changing, in all its terms, $Z$ into $A$. ... And we find in a similar manner the value of the other unknowns.
III. Students use Cramer's method of derangements to determine the same formulas for the cases $n=2,3,4$.

## Alternating Symmetric Functions

IV. Cauchy considers multivariable functions in sets of variables of the form $a_{1}, a_{2}, \ldots, a_{n} ; b_{1}, b_{2}, \ldots, b_{n} ; \ldots$, in which transposing any one pair of indices changes only the sign of the value of the function. These he calls alternating symmetric functions. He shows that arbitrary permutations performed upon these indices also produces either no change or a change of sign depending on the parity of the permutation (even or odd):
"a certain kind of alternating symmetric function which present themselves in a great number of analytic investigations. It is by means of these functions that we express the general values of unknowns that many equations of the first degree contain. ...Mr. Gauss has labeled these same functions with the name determinants."

## Alternating Symmetric Functions

- He calls a symmetric system any array in the $n^{2}$ quantities $a_{i, j}$, what we today call an $n \times n$ matrix ( $a_{i j}$ ).
- Uses the notation $\mathbf{S}\left( \pm a_{1,1} a_{2,2} \ldots a_{n, n}\right)$ to denote the determinant of the symmetric system $\left(a_{i j}\right) . \mathbf{S}( \pm)$ acts as an antisymmetrizing operator on the second set of indices of the indicative term $a_{1,1} a_{2,2} \ldots a_{n, n}$. Thus, it denotes the (now familiar) sum

$$
\sum_{\text {perms } \sigma} \operatorname{sign}(\sigma) \cdot a_{1, \sigma(1)} a_{2, \sigma(2)} \ldots a_{n, \sigma(n)}
$$

- Students are asked to tabulate the $n$ ! terms ( $n=2,3,4$ ) of $\mathbf{S}\left( \pm a_{1,1} a_{2,2} \ldots a_{n, n}\right)$ and determine the parities of the summands.
- Cauchy then shows that the parity also equals $n-g$, where $g=$ the number of disjoint cycles in a representation of the permutation; students recalculate using this method.


## Alternating Symmetric Functions

V. As homework, students pause reading Cauchy to compare the modern methods for computing $2 \times 2$ and $3 \times 3$ determinants \& determinants of upper triangular matrices.

- Next, consideration of the action of the operator $\mathbf{S}( \pm)$ allows Cauchy to conclude that $\operatorname{det}\left(a_{i j}\right)^{T}=\operatorname{det}\left(a_{i j}\right)$.
- Then describes what we know today as Laplace expansion by noting that the determinant is also given by the formula

$$
\mathbf{S}\left[ \pm a_{\mu, \mu} \mathbf{S}\left( \pm a_{1,1} a_{2,2} \ldots a_{\mu-1, \mu-1} a_{\mu+1, \mu+1} \ldots a_{n-1, n-1}\right)\right]
$$

which he writes in the form

$$
a_{1, \mu} b_{1, \mu}+a_{2, \mu} b_{2, \mu}+\cdots+a_{\mu, \mu} b_{\mu, \mu}+\cdots+a_{n, \mu} b_{n, \mu}
$$

VI. Students work out the complicated notion of expansion along a row or column.

## Alternating Symmetric Functions

VII. In a second pause from reading Cauchy, students (for homework) consider the effect on the determinant of interchanging a row or column of the underlying matrix.

- From this, it is a corollary that any matrix with a duplicate row or column has zero determinant.
- The symmetric system $\left(b_{j, i}\right)$ he calls "adjoint to the system ( $a_{i, j}$ )."
VIII. Cauchy then derives what in modern terms is the adjoint formula: $A \cdot \operatorname{adj} A=(\operatorname{det} A) \cdot I_{n}$, from which he obtains a proof of Cramer's Rule!


## Alternating Symmetric Functions

IX. Finally, Cauchy develops the composition of symmetric systems, i.e., the multiplication of matrices: $m_{i, j}$ is a system whose component systems are $a_{i, j}$ and $\alpha_{i, j}$ (in modern form, $\left.\left(a_{i, j}\right)\left(\alpha_{j, k}\right)=\left(m_{i, k}\right)\right)$ whenever

$$
a_{\nu, 1} \alpha_{\mu, 1}+a_{\nu, 2} \alpha_{\mu, 2}+\cdots+a_{\nu, n} \alpha_{\mu, n}=m_{\mu, \nu}
$$

- His last trick is to demonstrate the product formula for determinants: where $M_{n}, D_{n}, \delta_{n}$ are the respective determinants of the systems $m_{i, j}, a_{i, j}$ and $\alpha_{i, j}$, he shows that $M_{n}=D_{n} \delta_{n}$.


## Thanks for your attention!

Questions?

