Teaching Linear Algebra through its Applications

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A Simple Problem

- Find $u(x)$ that satisfies

$$-u''(x) = 4 \quad \forall x \in (-2, 2),$$
$$u(-2) = 0 \quad u(2) = 0.$$

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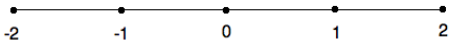
- Let's find a good approximation of $u(x)$ by using finite element methods.

A Neat Vector Space

- The mesh (split) of the domain $[-2, 2]$:



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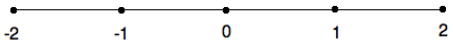

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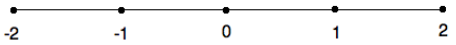
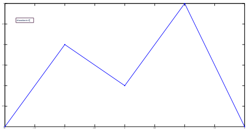
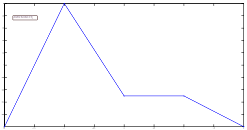
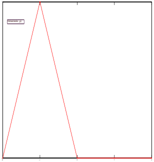


- Construct a vector space V_h associated with this mesh.
- $w_h \in V_h$ if and only if
  1. $w_h(x)$ is continuous on $[-2, 2]$ and $w_h(-2) = 0 = w_h(2)$.
  2. $w_h(x)$ is piecewise linear on all subintervals in the mesh.

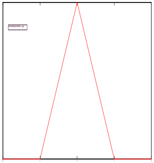




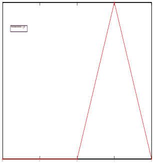
A Basis for V_h



(a) $\phi_1(x)$



(b) $\phi_2(x)$

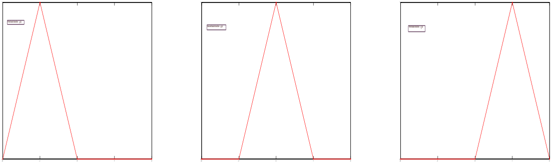


(c) $\phi_3(x)$

$$\phi_i(v_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

- $\{\phi_1(x), \phi_2(x), \phi_3(x)\}$ form a basis of V_h.

A Basis for V_h



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     $c_1\phi_1(x) + c_2\phi_2(x) + 3\phi_3(x) = 0 \rightarrow c_1 = c_2 = c_3 = 0$

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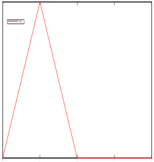

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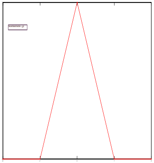

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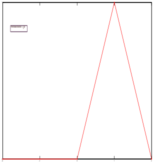

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- Let $w(x)$ be a “nice” function that satisfies $w(-2) = 0 = w(2)$.
- Find $u(x)$ such that

$$\int_{-2}^{2} u'(x)w'(x)dx = \int_{-2}^{2} 4w(x)dx \quad \forall w(x)$$
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- The Discrete Problem: Find $u_h(x) \in V_h$ that satisfies

$$\int_{-2}^{2} u_h'(x)w_h'(x)dx = \int_{-2}^{2} 4w_h(x)dx \qquad \forall w_h(x) \in V_h.$$

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$$Ac = b \qquad 3 \times 3 \text{ Matrix System}$$

$$A(i,j) = \int_{-2}^{2} \phi_j'(x) \phi_i'(x) dx \qquad \forall i, j = 1, 2, 3$$

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- Solve $Ac = b \Rightarrow u_h(x) = \sum_{j=1}^{3} c_j \phi_j(x)$

Finding the Finite Element Approximation $u_h(x)$

- $$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} \quad c = \begin{bmatrix} 6 \\ 8 \\ 6 \end{bmatrix}$$

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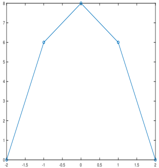

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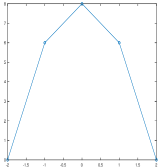
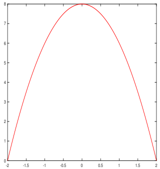


- Exact Solution $u(x) = -2x^2 + 8$



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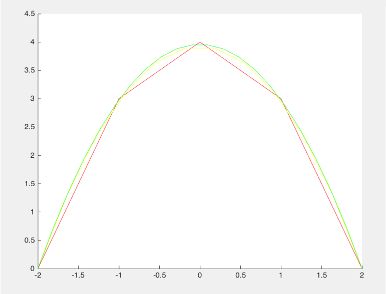
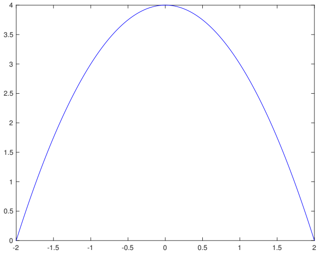
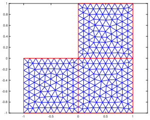
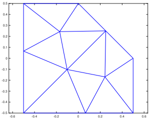
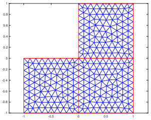
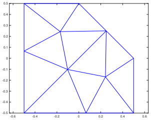
- Example
  FEM Approximations (left)   vs   Exact Solution (right)





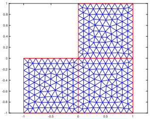
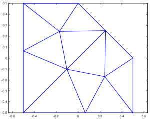




- Poisson equation: find $u(x, y)$ such that

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega = (-1,1) \times (-1,1), \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

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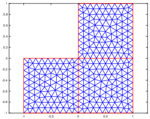
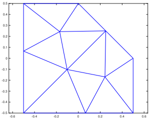

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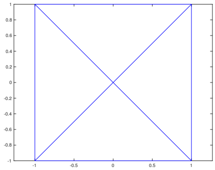
Finite Element Methods in Two-Dimensions

- The mesh (triangulation) of the domain:



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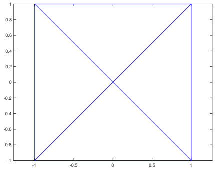

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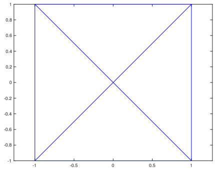

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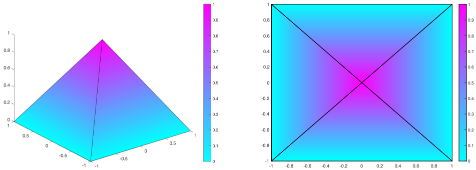

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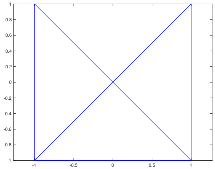

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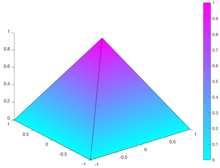
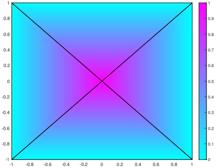
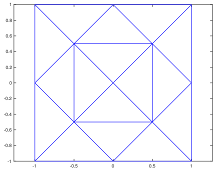



- Dimension of V_h for this mesh: 1

- Another mesh (triangulation) of the domain:



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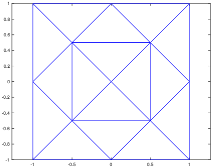

- Dimension of V_h for this mesh: 5

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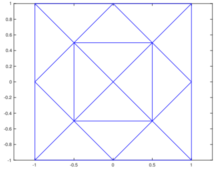
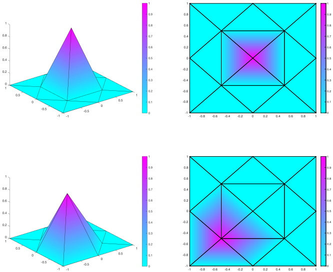


- Dimension of V_h for this mesh: 5
- Some basis function pictures of V_h:



- An alternative treatment to liver cancer.

FEMs for the Axisymmetric Maxwell Equations: Hepatic Microwave Ablation

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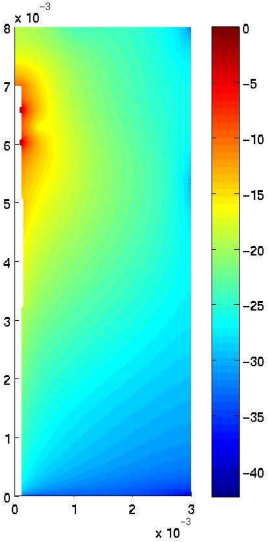
- Design of antennas : The electromagnetic power pattern should be highly localized near the tip of the antenna.
- Approximate the solution to the Axisymmetric Maxwell Equations to get an approximate electromagnetic power distribution for different antenna designs.

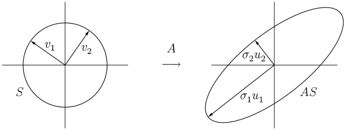
Electromagnetic Power Distribution for Antenna Design:



- Singular Value Decomposition

$$A_{m\times n} = U_{m\times m} \cdot \Sigma_{m\times n} \cdot V^T_{n\times n}$$

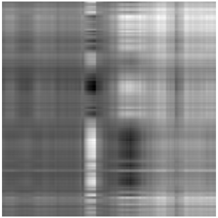
U and V: orthogonal matrices
Σ: "diagonal" matrix with diagonal entries
$\sigma_1 \geq \sigma_2 \geq ...\sigma_p \geq 0$, $p = min(m, n)$.



Singular Value Decomposition and Image Compression

$$A = \sigma_1 \cdot \vec{u_1} \cdot \vec{v_1}^T + \sigma_2 \cdot \vec{u_2} \cdot \vec{v_2}^T \cdots + \sigma_n \cdot \vec{u_n} \cdot \vec{v_n}^T$$

Singular Value Decomposition and Image Compression



Thank you!