Discovering Ill-Conditioned Systems Via Plane Deformations

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Who?

- Quantitative Analysis (Math: Algebra, Linear Systems, Calculus)
- B.S. in Business Administration
- Background: Algebra I and armed with TI83/84

Why?

- Quantitative Analysis for Management (Business).
- Play with vector and matrix algebra.

Intro Problem

$$209(0.5) + 472(0.5) = 340.5$$
$$56(0.5) + 125(0.5) = 90.5$$

$$209x + 472y = 340$$
$$56x + 125y = 90$$

Intro Problem

ORIGINAL

$$209x + 472y = 340.5$$
$$56x + 125y = 90.5$$

$$x = 0.5\ cup$$
$$y = 0.5\ cup$$

PERTURBED

$$209x + 472y = 340$$
$$56x + 125y = 90$$

$$x = -0.065\ cup$$
$$y = 0.749\ cup$$

Intro Problem

ORIGINAL

$$209x + 472y = 340.5$$
$$56x + 125y = 90.5$$

$$x = 0.5\ cup$$
$$y = 0.5\ cup$$

PERTURBED

$$209x + 472y = 339.5$$
$$56x + 125y = 90.25$$

$$x = 0.523\ cup$$
$$y = 0.488\ cup$$

Not so much the size, but the "direction" of change may make a difference:

$$\begin{pmatrix} -0.5 \\ -0.5 \end{pmatrix} \text{ versus } \begin{pmatrix} -1 \\ -0.25 \end{pmatrix}$$

1:1 change versus 4:1 change

The plan:

- Test 2×2 systems for ill-conditioning.
- Check if small perturbations of the RHS result in big perturbations to the solution.
- Do it with a TI83/84.

Test by perturbing the RHS in “all” directions:

ORIGINAL

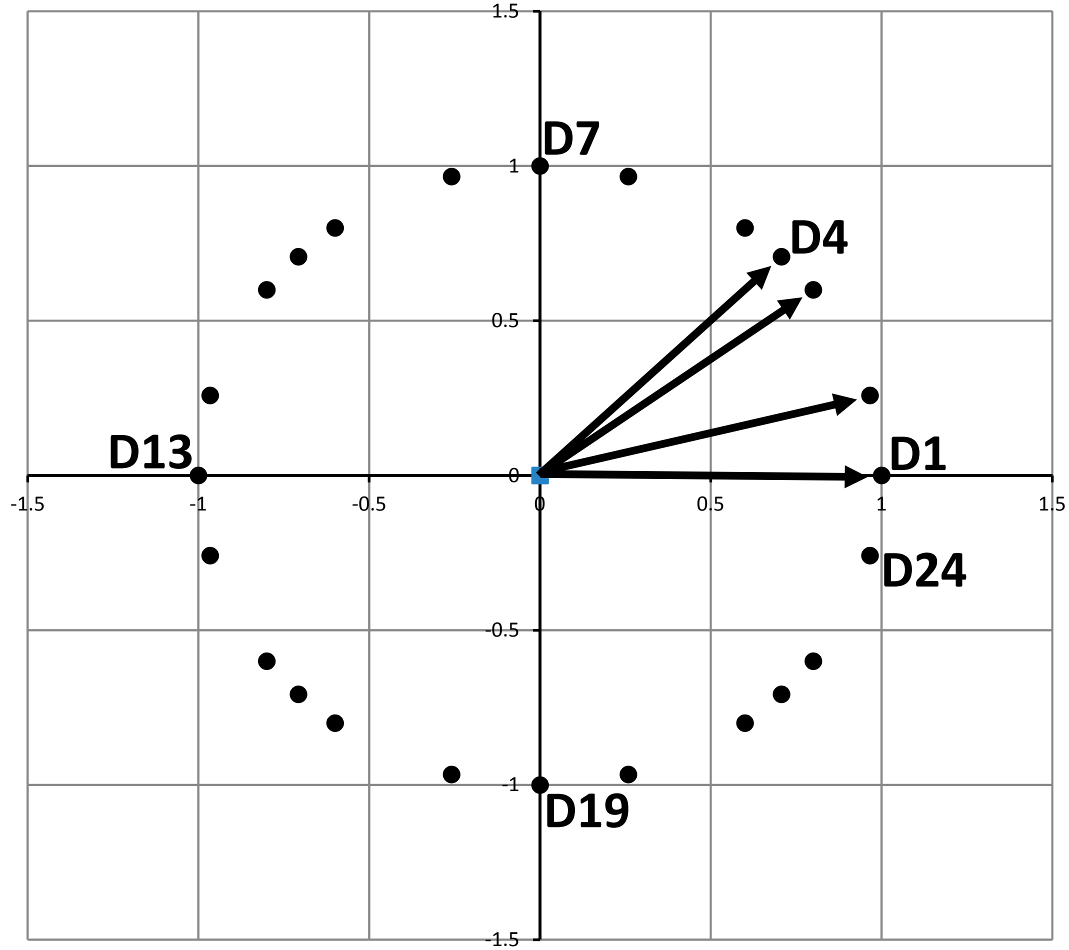
$$Ax = b$$

$$x0 = A^{-1}b$$

PERTURBED

$$Ax = b + \Delta b \Leftarrow$$

$$\Delta x = x - x0$$



D7
D4
D13
D1
D24
D19
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5

TI83/84:

- Enter the Δb's (=D's) in the matrix $[K]_{2\times 24} = \begin{bmatrix} \Delta b_x & \cdots \\ \Delta b_y & \end{bmatrix}$
- Create the matrix $[I]_{2\times 24} = \begin{bmatrix} 1 & \cdots \\ 0 & \end{bmatrix}$
- Create the matrix $[J]_{2\times 24} = \begin{bmatrix} 0 & \cdots \\ 1 & \end{bmatrix}$

TI83/84:

Enter the coefficient matrix $[A]$

For $b = \begin{pmatrix} b_x \\ b_y \end{pmatrix}$, calculate $[A]^{-1}\left(b_x[I] + b_y[J] + [K]\right)$

Analysis:

- Calculate the Δx's and their lengths.
- Identify the “longest” Δx and corresponding Δb.
- Identify the “shortest” Δx and corresponding Δb.
- Graph the Δx's to assess distance from the solution.

Example 1 (joint effort)

$$x + y = 100$$
$$8x - 2y = 100$$

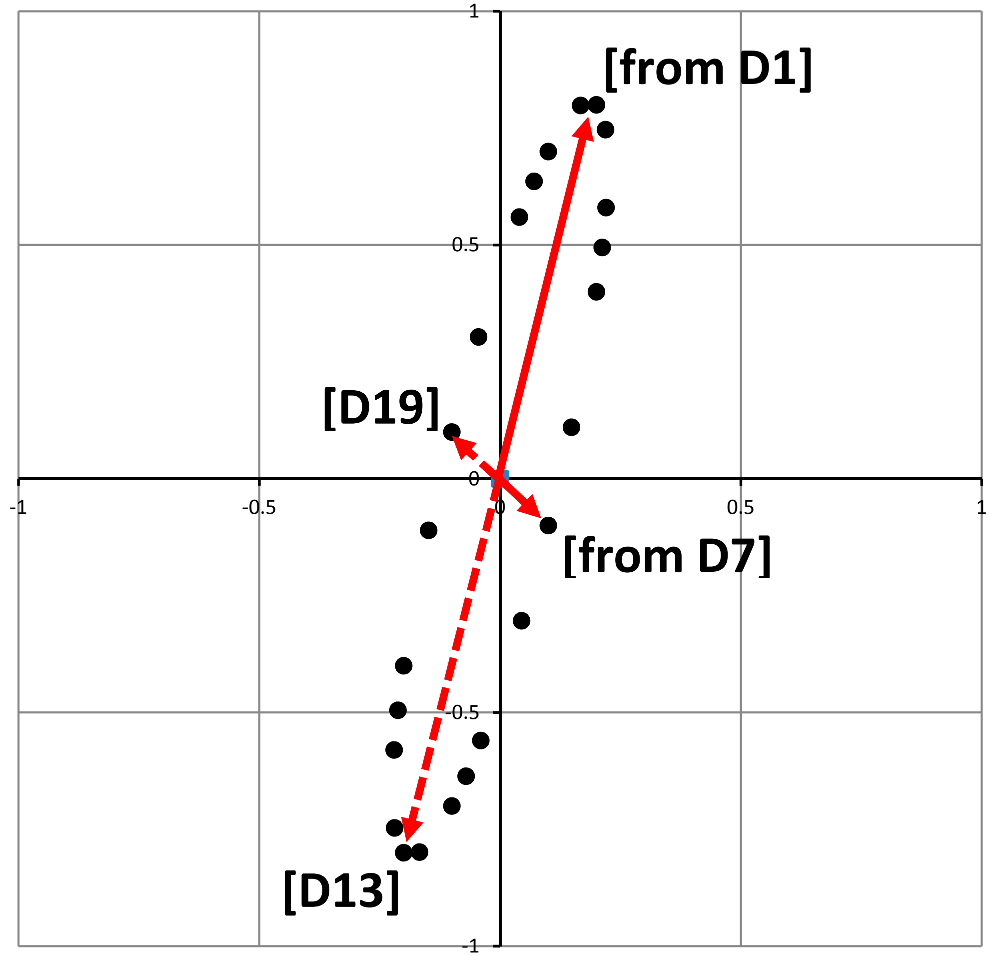
Example 1 $b = \binom{100}{100}$ $x0 = \binom{x}{y} = \binom{30}{70}$

Direction	Δx	Δy	length
D1	0.2	0.8	0.825
D2	0.219	0.747	0.778
D3	0.22	0.58	0.620
D4	0.212	0.495	0.539
*	*	*	*
D24	0.167	0.799	0.816

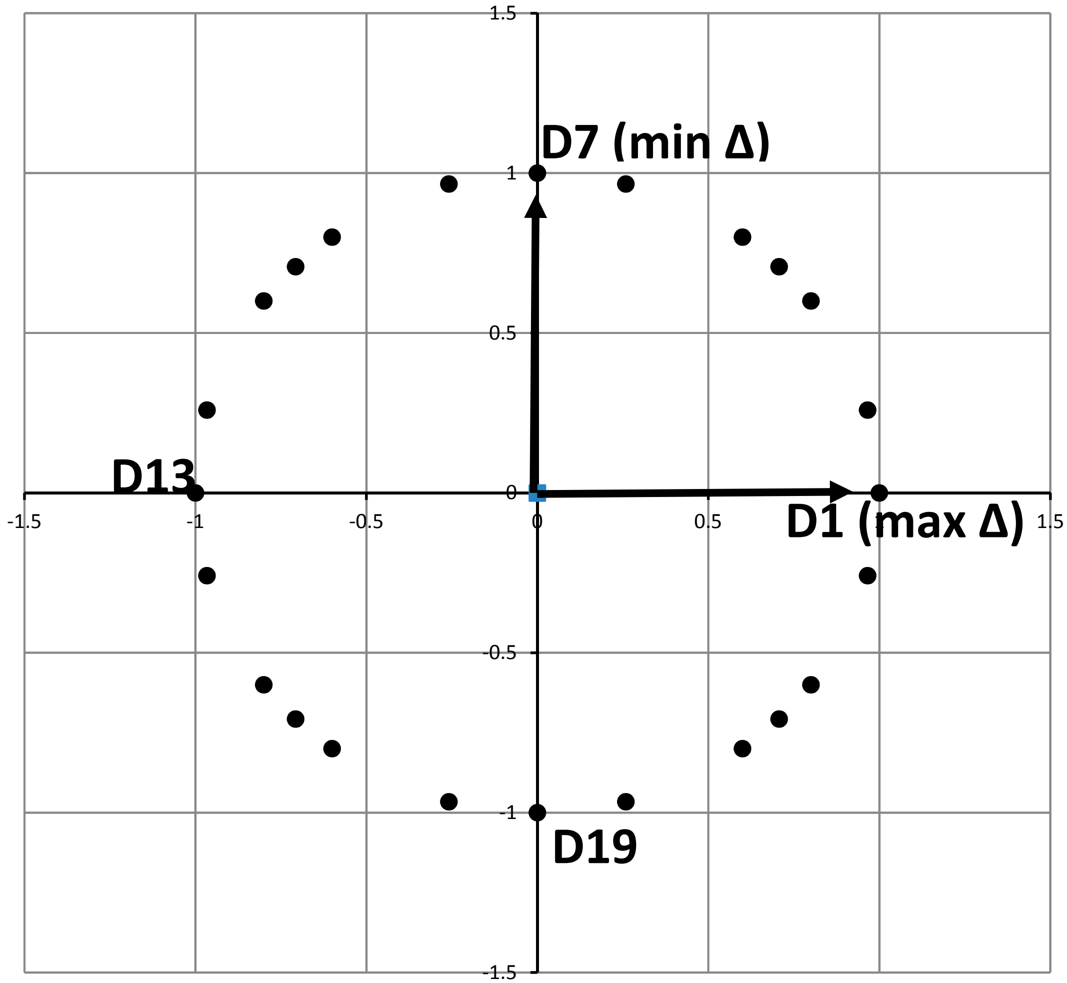
Example 1

$$b = \binom{100}{100} \qquad x0 = \binom{x}{y} = \binom{30}{70}$$

Direction	Δx	Δy	length
D1 *	0.2 *	0.8 *	0.825 (max) *
D7 *	0.1 *	-0.1 *	0.141 (min) *



[from D1]
[D19]
[from D7]
[D13]
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1



D7 (min Δ)
D13
D1 (max Δ)
D19
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1.5

Business Result

For $b = \begin{pmatrix} 100 \\ 100 \end{pmatrix}$

- 0.7% change in b.
- Up to 1.1% change in the solution.

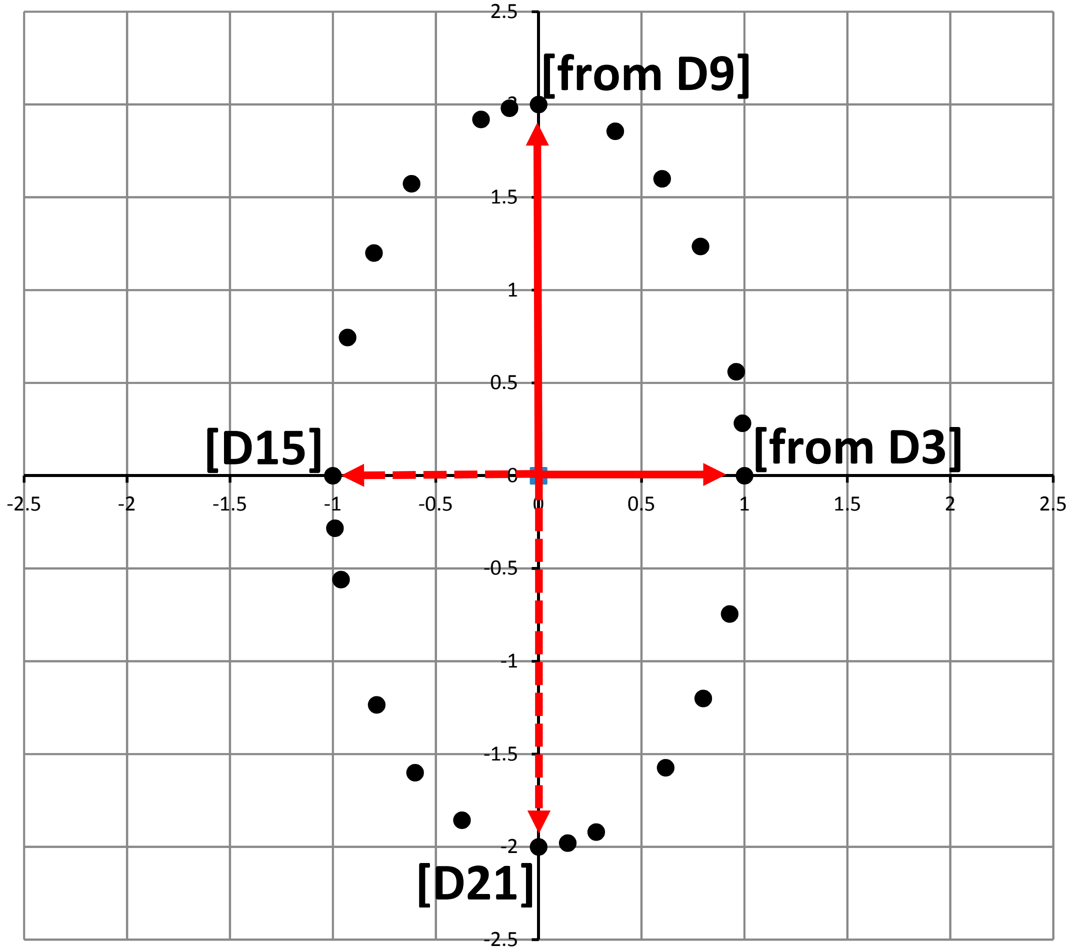
Example 2

$$0.8x - 0.3y = 45$$
$$0.6x + 0.4y = 35$$

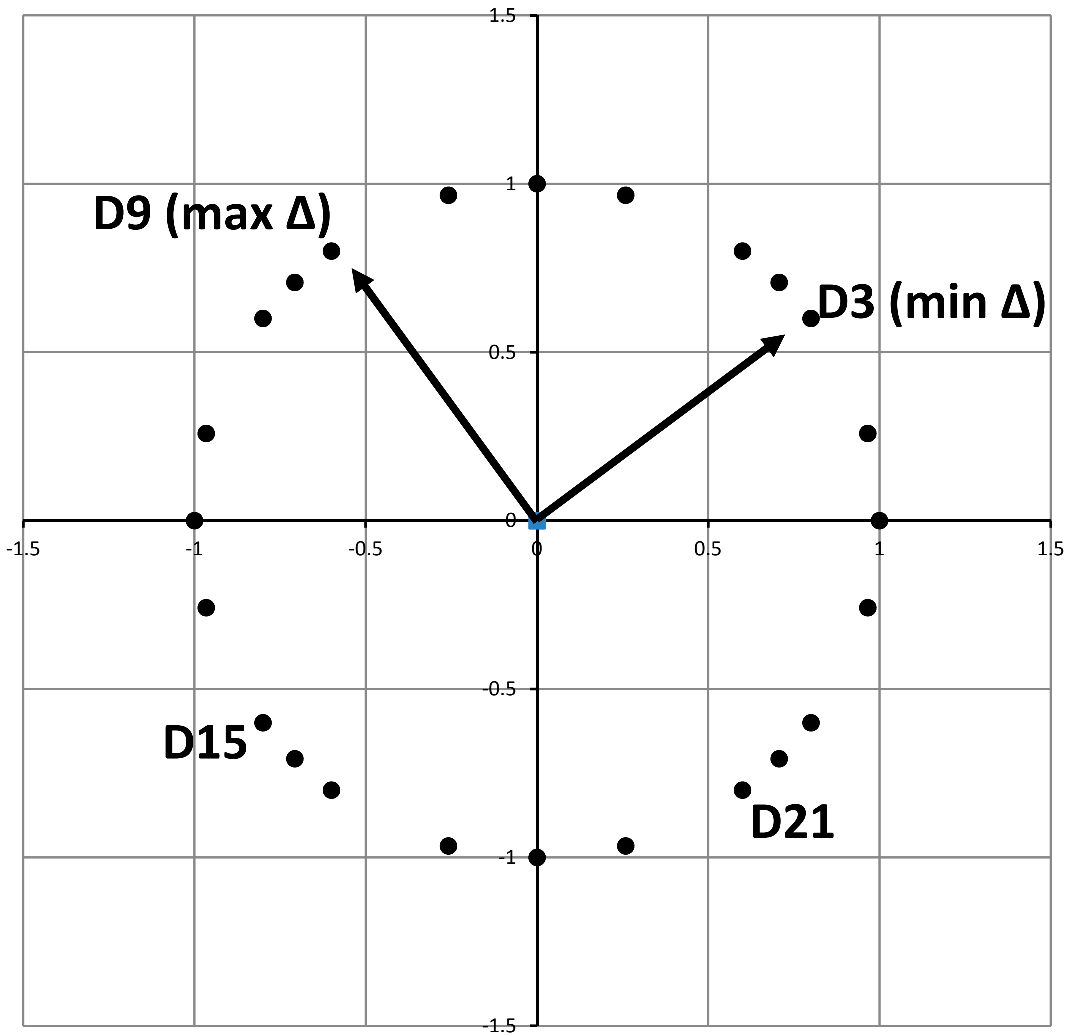
Example 2

$b = \binom{45}{35}$ $x0 = \binom{x}{y} = \binom{57}{2}$

Direction	Δx	Δy	length
*	*	*	*
D3	1	0	1 (min)
*	*	*	*
D9	0	2	2 (max)
*	*	*	*



[from D9]
[D15]
[from D3]
[D21]
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5


D9 (max Δ)
D3 (min Δ)
D15
D21
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5

Business Result

For $b = \begin{pmatrix} 45 \\ 35 \end{pmatrix}$

- 1.8% change b.
- Up to 3.5% change in the solution.

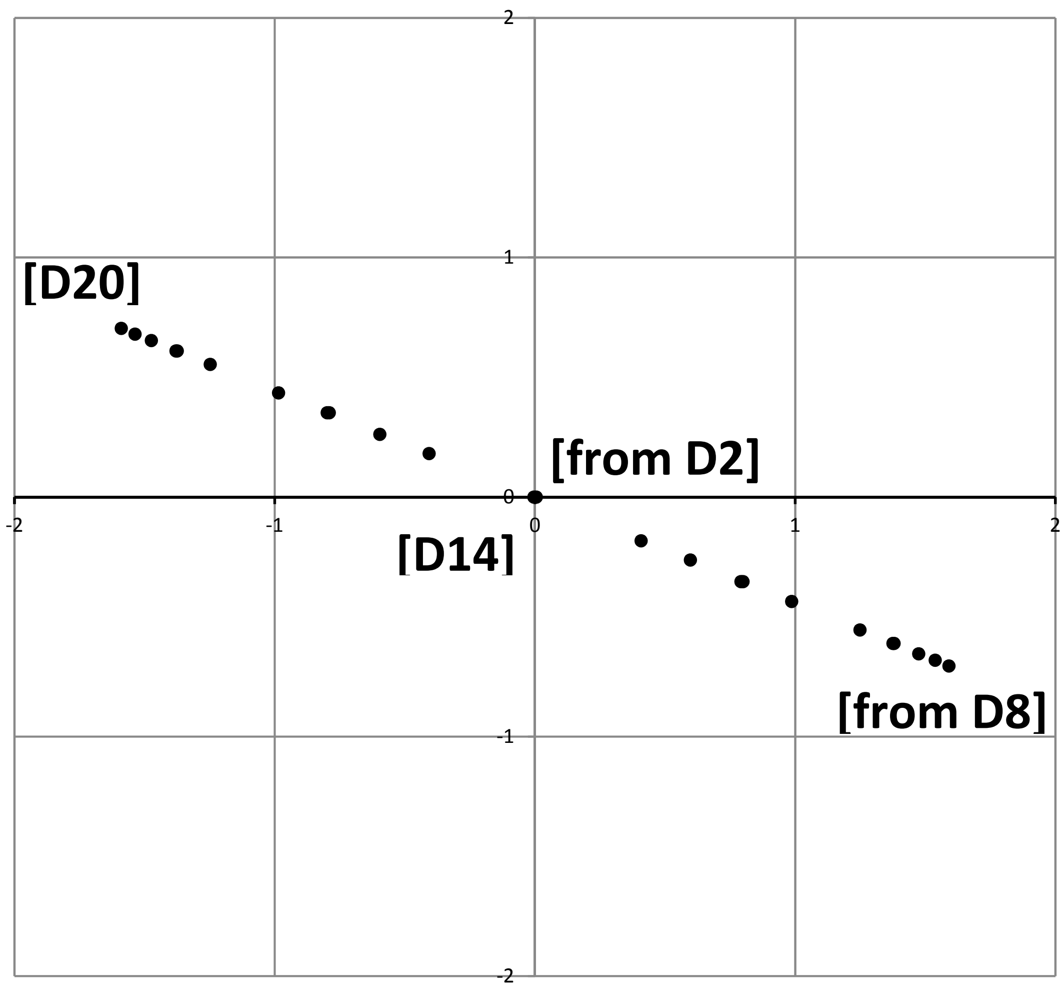
Example 3

$$209x + 472y = 340.5$$
$$56x + 125y = 90.5$$

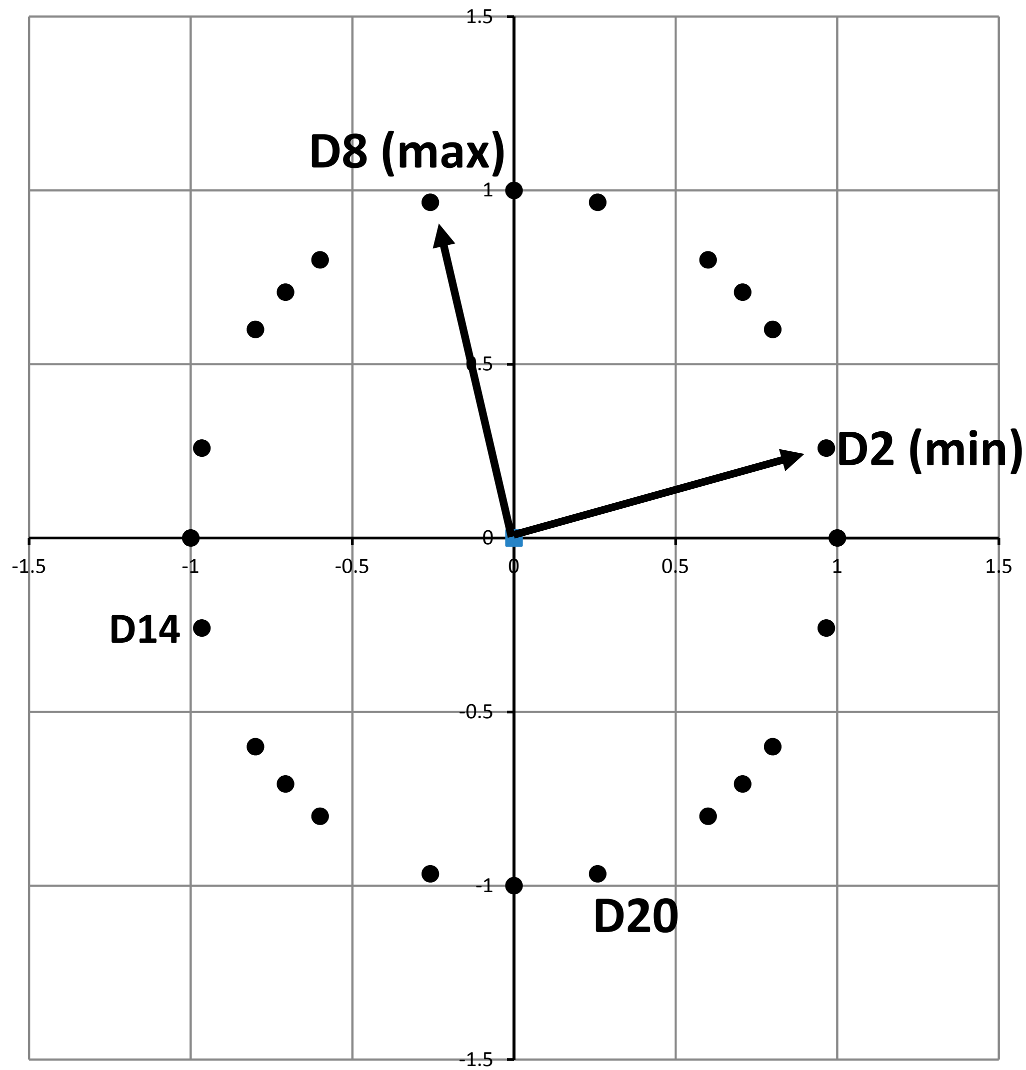
Example 3

$$b = \begin{pmatrix} 340.5 \\ 90.5 \end{pmatrix} \qquad x0 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

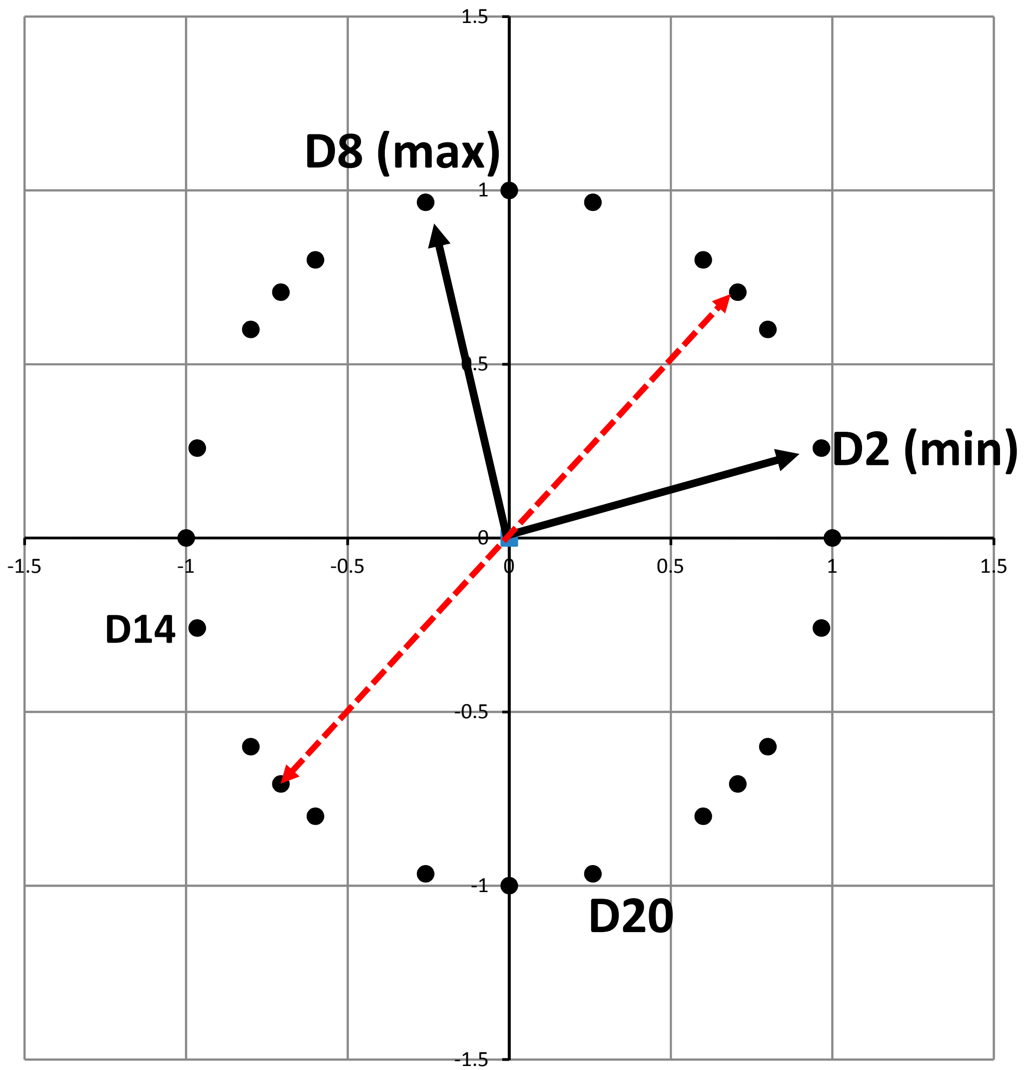
Direction	Δx	Δy	length
*	*	*	*
D2	0.005	0.000	0.005 (min)
*	*	*	*
D8	1.590	-0.705	1.740 (max)
*	*	*	*



[D20]
[from D2]
[D14]
[from D8]
2
1
0
-1
-2
-2
-1
0
1
2



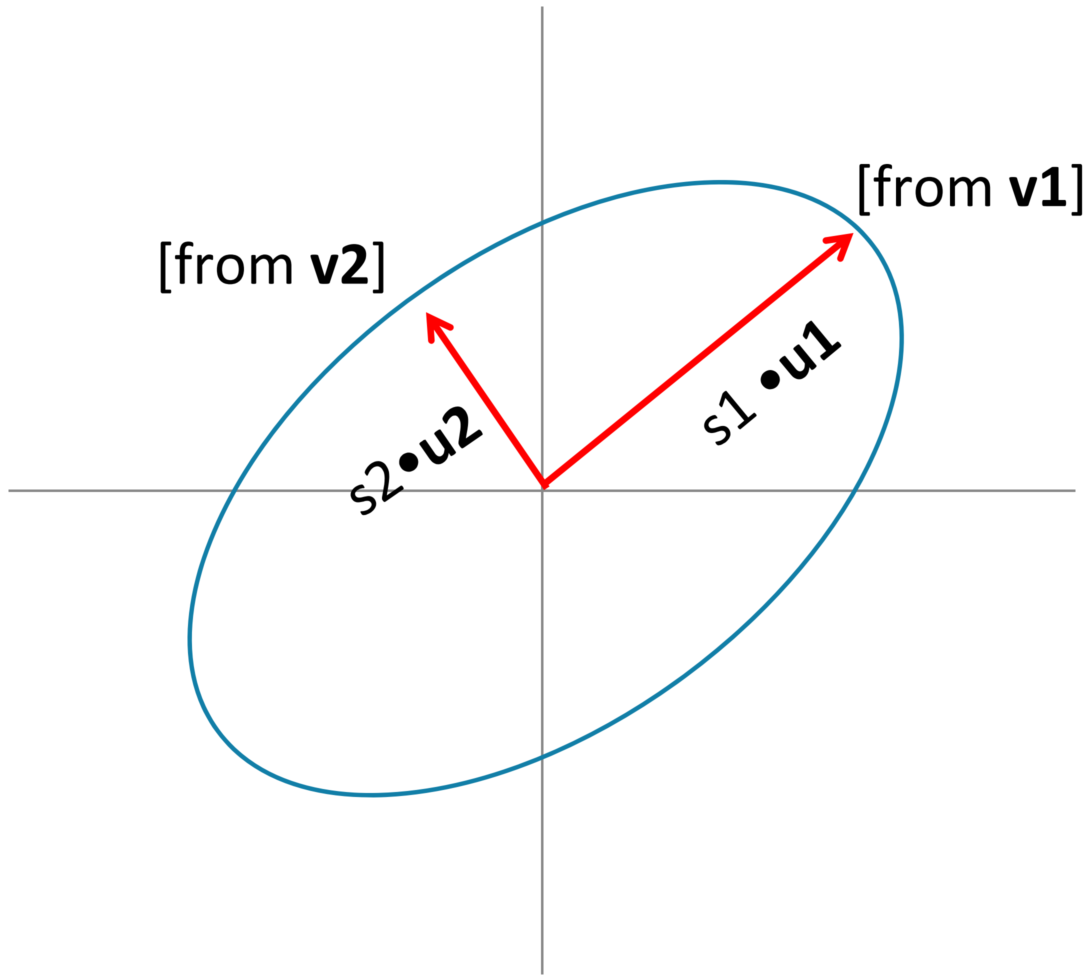
D8 (max)
D2 (min)
D14
D20
1.5
1
0.5
0
-0.5
-1
-1.5


D8 (max)
D2 (min)
D14
D20
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5

Business Result

For $b = \begin{pmatrix} 340.5 \\ 90.5 \end{pmatrix}$

- 0.2% change in b.
- Up to 87.4% change in the solution.


[from **v1**]
[from **v2**]
s1 •**u1**
s2 •**u2**

If you have the unit vectors that map to the vectors representing the major and minor axes,

$$Av_1 = s_1 u_1$$
$$Av_2 = s_2 u_2$$

then you can construct A via the Singular Value Decomposition (SVD)

Singular Value Decomposition (SVD)

$$A = USV^t$$

$$A = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix}^t$$

Did they get it? Yes!

- No problems with matrix manipulations and technology.
- Stronger students wanted to do more.

Did they get it? Yes!

- Graphing was the big hurdle.
- Students had trouble with the “business result.”

Next Time:

- More perturbations?
- Supply the graph?
- Forward Problem (Δx) or Backward Problem (Δb)?