Two True/False Questions on Linear Independence and an Application to a Set Theory Problem

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Support the Theme:

A beginning Linear Algebra course provides excellent opportunities to introduce inexperienced students to mathematical thinking and problem solving.

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  - no experience in writing proofs or solving real problems;
- Textbook:
  - Anton and Rorres: *Elementary Linear Algebra, Applications Version.*
  - selected due to its content, approach and assignments.

CONTENTS

Data files for exercises requiring MATLAB, Mathematica and Maple can be found on the Student Companion Site

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True-False Questions

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True-False Questions

- on homework and tests; either prove or disprove a statement with an argument or a counterexample;
- students found them challenging and interesting;
- an effective tool to realize many of the course goals.

True/False Question on Linear Independence/Dependence

A T/F question from homework (Exercise 4.3 in the textbook):

If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent nonzero vectors, then at least one vector $\mathbf{v}_k$ is a unique linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$.

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Definition from the textbook: a set of 2 or more vectors is linearly dependent if at least one is a linear combination of the others.

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Definition from the textbook: a set of 2 or more vectors is linearly dependent if at least one is a linear combination of the others.
Reasoning: flawed. typical for students at this level:

- ignoring $\mathbf{v}_k$ expressed as a linear combination of the previous vectors: $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$.
- ignoring uniqueness.
- ignoring the condition "nonzero", used or not.

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Intervention: make sure the meaning of the statement is understood.

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Intervention: make sure the meaning of the statement is understood.

Second reaction: It's false!

- insufficient understanding of linear dependence/independence;
- jump to conclusions based on wrong intuition;
- little or almost no evidence.

Class Discussion

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- Hint: use the equivalent statement of linearly dependent: $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is linearly dependent $\Rightarrow$ there are coefficients $c_1, \ldots, c_n$ not all zero such that $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$.

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- An idea: let m be the largest index such that $c_m \neq 0$.
- Caution: take care of details: $m \leq n$, why? $m \geq 2$, why? justify and see how the conditions are used.

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- We have started from the entire set and considered the dependence of the subsets;
- What about starting from the beginning and considering the independence of the subsets?

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- starting with $\{\mathbf{v}_1\}$, is it independent/dependent, why? they see "nonzero" is used.
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- There is a transition, a moment when the sets first become dependent.
- Stating it mathematically, let k be the smallest index such that $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ is dependent;
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- Show the statement holds for this k.
- Remind them to justify that such a k exists, and find out its range: $2 \le k \le n$.

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        - work from what one knows, what one is familiar with;
        - if stuck, go back to the problem and analyze it more carefully;
        - see what the problem requires, not what's convenient for you or what you have decided to do.

Summary:
Apart from the content (independence/dependence) and technique (extremal argument), leading students through thinking and solving this problem gives them a taste of how one might approach a problem, analyze it, solve it and how to better organize a proof after reaching a rough argument. It also gives them the confidence that they *can* solve problems.

Collecting Examples and Problems

Examples and Problems that

- integrate fundamental concepts;
- illustrate style of arguments and introduce techniques;
- specific incidences $\rightarrow$ general(abstract) theories.

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e.g.,

- The Dimension Theorem

  for matrix transformations (linear transformations on R^n)

  $\downarrow$

  for general linear transformations

- Every n-dimensional vector space is isomorphic to R^n.
- Cauchy-Schwarz Inequality
  (for R^n, for general inner product spaces, and its connection to projection, linear dependence/independence)

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Utilizing such Examples and Problems

- intentional about exposing examples and assigning exercises throughout the course;
- let the students see the ideas in action in similar and diverse specific situations;
- general (abstract) observation would surface naturally and inevitably.

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A T/F question on a test:

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- good to be on the test after discussing general vector spaces:
  - concepts involved: dimension, span, subspaces, basis, etc.;
  - the key is a construction of such a basis;
  - the proof is a good practice of standard arguments about basis.

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- difficult for the students:
  - to understand the statement correctly, requires one to be clear about definitions and concepts;
  - not easy to arrive at a correct guess: most of them would have the wrong intuition;
  - situation is abstract (general), hard for them to get a grip on.

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Technique: Think in R^n to get an intuition.

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- Let V be R^2 and L be any line through origin, clearly there are bases of R^2 that contain no vectors from L the line.

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- Let V be R^2 and L be any line through origin, clearly there are bases of R^2 that contain no vectors from L the line.
- Observation in R^n suggests that the statement is True.
- Not easy for students to construct a basis in the general case.

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An observation from a related homework T/F question:

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- understand what it means for the statement to be True/False;
- not hard to arrive at a counterexample, a basis of P_4: $\{1 + x^4, x + x^4, x^2 + x^4, x^3 + x^4, x^4\}$;

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- not hard to arrive at a counterexample, a basis of P_4: $\{1 + x^4, x + x^4, x^2 + x^4, x^3 + x^4, x^4\}$;
- students feel the work is done once the problem is solved.

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- A: due to the standard basis.
- Q: how is the standard basis formed?
- A: by including vectors which are not in the span of the existing ones: $P_3 \subseteq P_4$; to grow a basis:

$$\{1, x, x^2, x^3\} \xrightarrow{\text{add } x^4} \{1, x, x^2, x^3, x^4\}$$

Push further, ask questions like why, how, and what about?

- Q: how about
  $\{1, x, x^2, x^3\} \longrightarrow \{1 + x^4, x + x^4, x^2 + x^4, x^3 + x^4, x^4\}$?

Push further, ask questions like why, how, and what about?

- Q: how about $\{1, x, x^2, x^3\} \longrightarrow \{1 + x^4, x + x^4, x^2 + x^4, x^3 + x^4, x^4\}$?
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- A little thinking leads to: translating by a vector which is not in the span of the existing ones.
- $\{1, x, x^2, x^3, x^4\}$ versus $\{1 + x^4, x + x^4, x^2 + x^4, x^3 + x^4, x^4\}$ two ways of growing a basis.

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A question on a test:

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Result:

- most realized that the statement is true, thinking in R^n or remembering the homework problem and class discussion, a few attempted to prove the statement in its generality.

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Result:

- most realized that the statement is true, thinking in R^n or remembering the homework problem and class discussion, a few attempted to prove the statement in its generality.
- more successful at impressing them that they could always think in R^n first—a way to approach and touch a problem, to think and develop intuition, based on what they know.

An application to a set theory question

A question about sets

The subsets $A_1, \ldots, A_k$ of $\{1, 2, \ldots, n\}$ are all different and such that $|A_i \cap A_j| = 1$ for $i \neq j$. Prove that $k \leq n$.

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- standard technique: translate it into a linear algebra problem of linear independence;
- a couple of nontrivial twists;

An application to a set theory question

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The subsets $A_1, \ldots, A_k$ of $\{1, 2, \ldots, n\}$ are all different and such that $|A_i \cap A_j| = 1$ for $i \neq j$. Prove that $k \leq n$.

- standard technique: translate it into a linear algebra problem of linear independence;
- a couple of nontrivial twists;
- a good exercise when considering the degenerate cases;

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- standard technique: translate it into a linear algebra problem of linear independence;
- a couple of nontrivial twists;
- a good exercise when considering the degenerate cases;
- good to assign as homework the "dual" problem:
  The subsets $A_1, \ldots, A_k$ of $[n] = \{1, 2, \ldots, n\}$ are different from $[n]$ and such that every pair of elements of $[n]$ is contained in exactly one A_j. Prove that $k \geq n$.

Thank You