

Two True/False Questions on Linear Independence and an Application to a Set Theory Problem

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Support the Theme:

A beginning Linear Algebra course provides excellent opportunities to introduce inexperienced students to **mathematical thinking** and **problem solving**.

Background

- Course: fundamental topics and general structures; balance theories and applications.

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- Learning Goals:
 - knowledge and knowhow;
 - to think mathematically, to investigate and solve problems;
 - important to pose good questions, kindle curiosities, explore connections and inspire interests.

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 - no experience in writing proofs or solving real problems;

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- Students:
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 - no experience in writing proofs or solving real problems;
- Textbook:
 - Anton and Rorres: *Elementary Linear Algebra, Applications Version*.
 - selected due to its content, approach and assignments.

CONTENTS

Data files for exercises requiring MATLAB, Mathematica and Maple can be found on the Student Companion

CHAPTER 1 Systems of Linear Equations and Matrices 1

- 1.1 Introduction to Systems of Linear Equations 2
- 1.2 Gaussian Elimination 11
- 1.3 Matrices and Matrix Operations 25
- 1.4 Inverses; Algebraic Properties of Matrices 39
- 1.5 Elementary Matrices and a Method for Finding A^{-1} 52
- 1.6 More on Linear Systems and Invertible Matrices 61
- 1.7 Diagonal, Triangular, and Symmetric Matrices 67
- 1.8 Matrix Transformations 75
- 1.9 Applications of Linear Systems 84
 - Network Analysis (Traffic Flow) 84
 - Electrical Circuits 86
 - Balancing Chemical Equations 88
 - Polynomial Interpolation 91
- 1.10 Application: Leontief Input-Output Models 96

CHAPTER 2 Determinants 105

- 2.1 Determinants by Cofactor Expansion 105
- 2.2 Evaluating Determinants by Row Reduction 113
- 2.3 Properties of Determinants; Cramer's Rule 118

CHAPTER 3 Euclidean Vector Spaces 131

- 3.1 Vectors in 2-Space, 3-Space, and n -Space 131
- 3.2 Norm, Dot Product, and Distance in \mathbb{R}^n 142
- 3.3 Orthogonality 155
- 3.4 The Geometry of Linear Systems 164
- 3.5 Cross Product 172

CHAPTER 4 General Vector Spaces 183

- 4.1 Real Vector Spaces 183
- 4.2 Subspaces 191
- 4.3 Linear Independence 202
- 4.4 Coordinates and Basis 212
- 4.5 Dimension 221
- 4.6 Change of Basis 229
- 4.7 Row Space, Column Space, and Null Space 237
- 4.8 Rank, Nullity, and the Fundamental Matrix Spaces 248
- 4.9 Basic Matrix Transformations in \mathbb{R}^2 and \mathbb{R}^3 259
- 4.10 Properties of Matrix Transformations 270
- 4.11 Application: Geometry of Matrix Operators on \mathbb{R}^2 280

CHAPTER 5 Eigenvalues and Eigenvectors 291

- 5.1 Eigenvalues and Eigenvectors 291
- 5.2 Diagonalization 302
- 5.3 Complex Vector Spaces 313
- 5.4 **Application:** Differential Equations 326
- 5.5 **Application:** Dynamical Systems and Markov Chains 332

CHAPTER 6 Inner Product Spaces 345

- 6.1 Inner Products 345
- 6.2 Angle and Orthogonality in Inner Product Spaces 355
- 6.3 Gram-Schmidt Process; QR -Decomposition 364
- 6.4 Best Approximation; Least Squares 378
- 6.5 **Application:** Mathematical Modeling Using Least Squares 387
- 6.6 **Application:** Function Approximation; Fourier Series 394

CHAPTER 7 Diagonalization and Quadratic Forms 401

- 7.1 Orthogonal Matrices 401
- 7.2 Orthogonal Diagonalization 409
- 7.3 Quadratic Forms 417
- 7.4 Optimization Using Quadratic Forms 429
- 7.5 Hermitian, Unitary, and Normal Matrices 437

CHAPTER 8 General Linear Transformations 447

- 8.1 General Linear Transformations 447
- 8.2 Compositions and Inverse Transformations 458
- 8.3 Isomorphism 466
- 8.4 Matrices for General Linear Transformations 472
- 8.5 Similarity 481

CHAPTER 9 Numerical Methods 491

- 9.1 LU -Decompositions 491
- 9.2 The Power Method 501
- 9.3 Comparison of Procedures for Solving Linear Systems 509
- 9.4 Singular Value Decomposition 514
- 9.5 **Application:** Data Compression Using Singular Value Decomposition 521

CHAPTER 10 Applications of Linear Algebra 527

- 10.1 Constructing Curves and Surfaces Through Specified Points 528
- 10.2 The Earliest Applications of Linear Algebra 533
- 10.3 Cubic Spline Interpolation 540

True-False Questions

- on homework and tests; either prove or disprove a statement with an argument or a counterexample;

True-False Questions

- on homework and tests; either prove or disprove a statement with an argument or a counterexample;
- students found them challenging and interesting;
- an effective tool to realize many of the course goals.

True/False Question on Linear Independence/Dependence

A T/F question from homework (Exercise 4.3 in the textbook):

If $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly dependent nonzero vectors, then at least one vector \mathbf{v}_k is a unique linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_{k-1}$.

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Reasoning: flawed. typical for students at this level:

- ignoring \mathbf{v}_k expressed as a linear combination of the previous vectors: $\mathbf{v}_1, \dots, \mathbf{v}_{k-1}$.
- ignoring uniqueness.
- ignoring the condition “nonzero”, used or not.

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Intervention: make sure the meaning of the statement is understood.

Second reaction: It's false!

- insufficient understanding of linear dependence/independence;
- jump to conclusions based on wrong intuition;
- little or almost no evidence.

True/False Question on Linear Independence/Dependence

Class Discussion

- Let them think and discuss;

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- Hint: use the equivalent statement of linearly dependent:
 $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is linearly dependent \Rightarrow there are coefficients c_1, \dots, c_n not all zero such that $c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n = \mathbf{0}$.

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- Question: what's good about having a non-zero coefficient?
If $c_m \neq 0$, then \mathbf{v}_m can be expressed as a linear combination of the others.

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- **An idea:** let m be the **largest** index such that $c_m \neq 0$.

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- **An idea:** let m be the **largest** index such that $c_m \neq 0$.
- Caution: take care of details: $m \leq n$, why? $m \geq 2$, why?
justify and see how the conditions are used.

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Suggestion:

- We have started from the entire set and considered the dependence of the subsets;

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- **Difficulty:** do not know $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{m-1}\}$ is independent;

Suggestion:

- We have started from the entire set and considered the dependence of the subsets;
- What about starting from the beginning and considering the independence of the subsets?

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- consider $\{\mathbf{v}_1, \mathbf{v}_2\}$, easy to see it can be independent or dependent;
- remembering the entire set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is dependent.

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An idea!

- There is a **transition**, a moment when the sets first become dependent.
- Stating it mathematically, let k be the **smallest** index such that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is dependent;
- Observe $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k-1}\}$ is independent;
- Show the statement holds for this k .

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- Observe $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k-1}\}$ is independent;
- Show the statement holds for this k .
- Remind them to justify that such a k exists, and find out its range: $2 \leq k \leq n$.

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Wrapping up:

- Complete the argument;

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- Complete the argument;
- Present in class and discuss;

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- Summarize and emphasize:

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 - Understand dependence/independence better: connection to existence/uniqueness.

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- Complete the argument;
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 - Extremal argument: extremal choice reveals more information.

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 - Same “trick” used elsewhere: e.g., proof of eigenvectors corresponding to distinct eigenvalues are independent.

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 - How would one think of doing it this way?

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 - **How would one think of doing it this way?**
 - experience;

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 - work from what one knows, what one is familiar with;

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 - **How would one think of doing it this way?**
 - experience;
 - work from what one knows, what one is familiar with;
 - if stuck, go back to the problem and analyze it more carefully;
 - see what the problem requires, not what’s convenient for you or what you have decided to do.

True/False Question on Linear Independence/Dependence

Summary:

Apart from the **content** (independence/dependence) and **technique** (extremal argument), leading students through thinking and solving this problem gives them a **taste** of how one might approach a problem, analyze it, solve it and how to better organize a proof after reaching a rough argument. It also gives them the **confidence** that they *can* solve problems.

Collecting Examples and Problems

Examples and Problems that

- integrate fundamental concepts;
- illustrate style of arguments and introduce techniques;
- specific incidences \rightarrow general(abstract) theories.

e.g.,

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e.g.,

- The Dimension Theorem

for matrix transformations (linear transformations on R^n)



for general linear transformations

- Every n -dimensional vector space is isomorphic to R^n .
- Cauchy-Schwarz Inequality
(for R^n , for general inner product spaces, and its connection to projection, linear dependence/independence)

Collecting Examples and Problems

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- integrate fundamental concepts;
- illustrate style of arguments and introduce techniques;
- specific incidences \rightarrow general (abstract) theories.

Utilizing such Examples and Problems

- intentional about exposing examples and assigning exercises throughout the course;
- let the students see the ideas in action in similar and diverse specific situations;
- general (abstract) observation would surface naturally and inevitably.

True/False Question on Basis

A T/F question on a test:

Let V be an n -dimensional vector space and L an m -dimensional subspace where $0 < m < n$. Then there is a basis for V such that it contains no vectors from L .

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A T/F question on a test:

Let V be an n -dimensional vector space and L an m -dimensional subspace where $0 < m < n$. Then there is a basis for V such that it contains no vectors from L .

- good to be on the test after discussing general vector spaces:
 - concepts involved: dimension, span, subspaces, basis, etc.;
 - the key is a construction of such a basis;
 - the proof is a good practice of standard arguments about basis.

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- difficult for the students:
 - to understand the statement correctly, requires one to be clear about definitions and concepts;
 - not easy to arrive at a correct guess: most of them would have the wrong intuition;
 - situation is abstract (general), hard for them to get a grip on.

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Technique: Think in R^n to get an intuition.

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- In the course: R^2 and $R^3 \rightarrow R^n \rightarrow$ general vector spaces
- Let V be R^2 and L be any line through origin, clearly there are bases of R^2 that contain no vectors from L the line.

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- Let V be R^2 and L be any line through origin, clearly there are bases of R^2 that contain no vectors from L the line.
- Observation in R^n suggests that the statement is True.
- Not easy for students to construct a basis in the general case.

True/False Question on Basis

An observation from a related homework T/F question:

Every basis of P_4 contains at least one polynomial of degree 3 or less. (P_n is the vector space of polynomials of degree at most n .)

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- familiar with standard basis of P_4 : $\{1, x, x^2, x^3, x^4\}$;

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- familiar with standard basis of P_4 : $\{1, x, x^2, x^3, x^4\}$;
- understand what it means for the statement to be True/False;
- not hard to arrive at a counterexample, a basis of P_4 :
 $\{1 + x^4, x + x^4, x^2 + x^4, x^3 + x^4, x^4\}$;

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 $\{1 + x^4, x + x^4, x^2 + x^4, x^3 + x^4, x^4\}$;
- students feel the work is done once the problem is solved.

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Push further, ask questions like why, how, and what about?

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- A: due to the standard basis.

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Push further, ask questions like why, how, and what about?

- Q: why the original false statement stated this way?
- A: due to the standard basis.
- Q: how is the standard basis formed?

True/False Question on Basis

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- Q: why the original false statement stated this way?
- A: due to the standard basis.
- Q: how is the standard basis formed?
- A: by including vectors which are not in the span of the existing ones: $P_3 \subseteq P_4$; to grow a basis:

$$\{1, x, x^2, x^3\} \xrightarrow{\text{add } x^4} \{1, x, x^2, x^3, x^4\}$$

True/False Question on Basis

Push further, ask questions like why, how, and what about?

- Q: how about

$$\{1, x, x^2, x^3\} \longrightarrow \{1 + x^4, x + x^4, x^2 + x^4, x^3 + x^4, x^4\}?$$

True/False Question on Basis

Push further, ask questions like why, how, and what about?

- Q: how about $\{1, x, x^2, x^3\} \longrightarrow \{1 + x^4, x + x^4, x^2 + x^4, x^3 + x^4, x^4\}$?
- **A little thinking leads to:** translating by a vector which is not in the span of the existing ones.

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- Q: how about $\{1, x, x^2, x^3\} \longrightarrow \{1 + x^4, x + x^4, x^2 + x^4, x^3 + x^4, x^4\}$?
- **A little thinking leads to:** translating by a vector which is not in the span of the existing ones.
- $\{1, x, x^2, x^3, x^4\}$ versus $\{1 + x^4, x + x^4, x^2 + x^4, x^3 + x^4, x^4\}$
two ways of growing a basis.

True/False Question on Basis

A question on a test:

Let V be an n -dimensional vector space and L an m -dimensional subspace where $0 < m < n$. Then there is a basis for V such that it contains no vectors from L .

Result:

- most realized that the statement is true, thinking in R^n or remembering the homework problem and class discussion, a few attempted to prove the statement in its generality.

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- most realized that the statement is true, thinking in R^n or remembering the homework problem and class discussion, a few attempted to prove the statement in its generality.
- more successful at impressing them that they could always think in R^n first—a way to approach and touch a problem, to think and develop intuition, based on what they know.

An application to a set theory question

A question about sets

The subsets A_1, \dots, A_k of $\{1, 2, \dots, n\}$ are all different and such that $|A_i \cap A_j| = 1$ for $i \neq j$. Prove that $k \leq n$.

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- standard technique: translate it into a linear algebra problem of linear independence;
- a couple of nontrivial twists;
- a good exercise when considering the degenerate cases;
- good to assign as homework the “dual” problem:
The subsets A_1, \dots, A_k of $[n] = \{1, 2, \dots, n\}$ are different from $[n]$ and such that every pair of elements of $[n]$ is contained in exactly one A_j . Prove that $k \geq n$.

Thank You