



Redmond

The TI-Nspire in Linear Algebra

Some of the usual things we do in Linear Algebra over the fields

$$\boxed{\mathbb{R}} \quad \text{and} \quad \boxed{\mathbb{C}}$$

for which the TI-Nspire has beautiful, built-in programs.

1. Solve:
$$\begin{cases} x + 2y + z = 1 \\ 4x + y + z = 5 \\ 6x + 5y + 4z = 5 \end{cases}$$

2. Compute:
$$\det \begin{bmatrix} 1 & 2 & 1 \\ 4 & 1 & 1 \\ 6 & 5 & 4 \end{bmatrix}$$

3. Compute:
$$\begin{bmatrix} 1 & 2 & 1 \\ 4 & 1 & 1 \\ 6 & 5 & 4 \end{bmatrix}^{-1}$$

4. Find a basis of $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 4 \\ 6 \end{bmatrix} \right\}$

5. Find Nullspace of :
$$\begin{bmatrix} 1 & 2 & 1 \\ 4 & 1 & 1 \\ 6 & 5 & 4 \end{bmatrix}$$

6. Find Range of :
$$\begin{bmatrix} 1 & 2 & 1 \\ 4 & 1 & 1 \\ 6 & 5 & 4 \end{bmatrix}$$

7. Find LU decomposition of :
$$\begin{bmatrix} 1 & 2 & 1 \\ 4 & 1 & 1 \\ 6 & 5 & 4 \end{bmatrix}$$

8. Diagonalize :
$$\begin{bmatrix} 1 & 6 & 6 \\ 4 & 1 & 1 \\ 5 & 3 & 3 \end{bmatrix}$$

The Finite Fields used in my Linear Algebra classes

\mathbb{F}_2

$$\mathbb{F}_2 = \{0, 1\}$$

+	0	1
	0	1
0	0	1
1	1	0

·	0	1
	0	0
0	0	0
1	0	1

\mathbb{F}_4

$$\mathbb{F}_4 = \{0, 1, a, b\}$$

+	0	1	a	b
	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

·	0	1	a	b
	0	0	0	0
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

\mathbb{F}_7

$$\mathbb{F}_7 = \{0, 1, 2, 3, 4, 5, 6\}$$

+	0	1	2	3	4	5	6
	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

·	0	1	2	3	4	5	6
	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Why Finite Fields?

- (1) It is fun!
- (2) It exposes non-math majors to worlds other than \mathbb{R} and \mathbb{C} .
- (3) Shows them similarities and differences.

The Programs created for the Finite Fields \mathbb{F}_2 , \mathbb{F}_4 , \mathbb{F}_7

1. Simplification

s 2, s 4, s 7

2. Matrix inverse

mi 2, mi 4, mi 7

3. Determinant

det 2, det 4, det 7

4. Factoring a polynomial

factor 2, factor 4, factor 7

5. Row reduction

rref 2, rref 4, rref 7

6. Row reduction (steps)

mrowadd 2, mrow 2, rowswap
mrowadd 4, mrow 4
mrowadd 7, mrow 7

7. LU decomposition

lup 2, lup 4, lup 7

The simplification functions **s2**, **s4**, **s7**

The simplification functions **s2**, **s4**, **s7** are very flexible. They simplify **any** linear combination from **any** of the vector spaces:

$$\boxed{\mathbb{F}^n} \quad \boxed{M_{n \times m}(\mathbb{F})} \quad \boxed{P_n(\mathbb{F})}$$

where $\mathbb{F} = \mathbb{F}_2, \mathbb{F}_4$ or \mathbb{F}_7

For example:

$s7\left(3 \cdot \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + 4 \cdot \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix} + 5 \cdot \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}\right)$	$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$
$s7\left(3 \cdot \begin{bmatrix} 4 & 5 \\ 1 & 6 \end{bmatrix} + 4 \cdot \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix} + 6 \cdot \begin{bmatrix} 6 & 5 \\ 5 & 4 \end{bmatrix}\right)$	$\begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$
$s7\left(5 \cdot (3 \cdot t^2 + 4 \cdot t + 1) + 6 \cdot (4 \cdot t^2 + 2 \cdot t + 5) + 4 \cdot (t + 5)\right)$	$4 \cdot t^2 + t + 6$
$s4\left(a \cdot \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} + b \cdot \begin{bmatrix} 1 \\ a \\ a \end{bmatrix} + \begin{bmatrix} b \\ 1 \\ b \end{bmatrix}\right)$	$\begin{bmatrix} b \\ 1 \\ 0 \end{bmatrix}$
$s4\left(b \cdot \begin{bmatrix} a & b & 1 \\ 0 & 1 & b \end{bmatrix} + a \cdot \begin{bmatrix} b & b & 1 \\ 1 & a & a \end{bmatrix}\right)$	$\begin{bmatrix} 0 & b & 1 \\ a & 0 & 1 \end{bmatrix}$
$s4\left(a \cdot (b \cdot t^2 + a \cdot t + 1) + b \cdot (t^2 + a) + b \cdot t^2 + t\right)$	$t^2 + a \cdot t + b$

These functions essentially first check if the entries are matrices or polynomials, and simplify accordingly.

Let's look at some examples of the functions mentioned before.

Examples

1. Let's solve $\begin{cases} x + y + z = 5 \\ y + 2z = 3 \\ x + 2y + 3z = 1 \end{cases}$ over \mathbb{F}_7 using **rref7**

The calculator screen shows the following steps:

- extract** $\begin{pmatrix} x+y+z=5 \\ y+2\cdot z=3 \\ x+2\cdot y+3\cdot z=1 \end{pmatrix}, \{x,y,z\}$
- Resulting augmented matrix: $\begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 1 \end{bmatrix}$
- rref7** $\begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 1 \end{pmatrix}$
- Resulting row echelon form: $\begin{bmatrix} 1 & 0 & 6 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

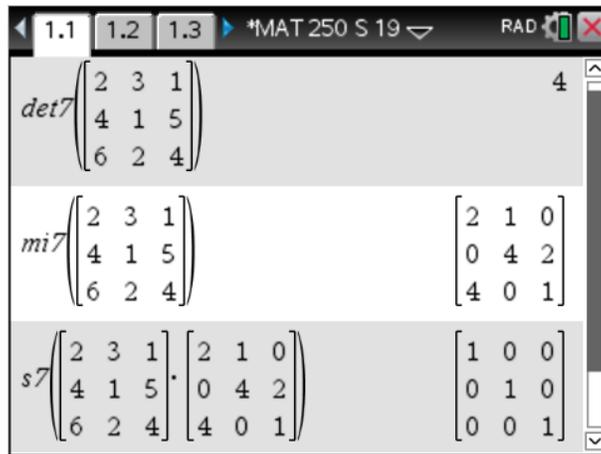
Hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$. Let's check these are indeed solutions:

The calculator screen shows the following verification steps:

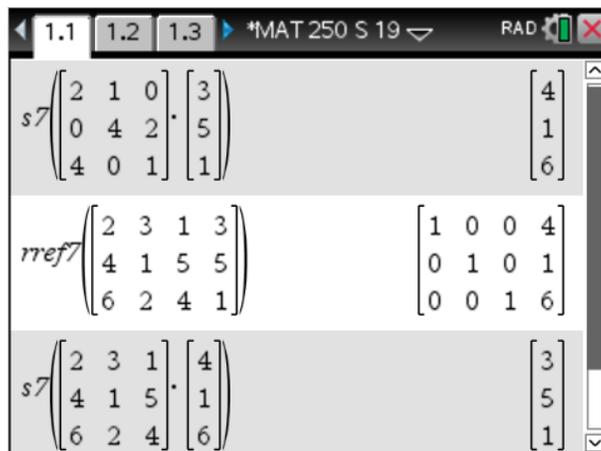
- s7** $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \cdot \left(\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} \right)$ Result: $\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$
- s7** $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} | t=4$ Result: $\begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}$
- s7** $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \cdot \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}$ Result: $\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$

Note this system of equations is **not** solvable over \mathbb{R} . Over \mathbb{F}_7 it has exactly 7 solutions.

2. Compute $\det \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 5 \\ 6 & 2 & 4 \end{bmatrix}$ over \mathbb{F}_7 . If it is non-zero, compute the inverse matrix.



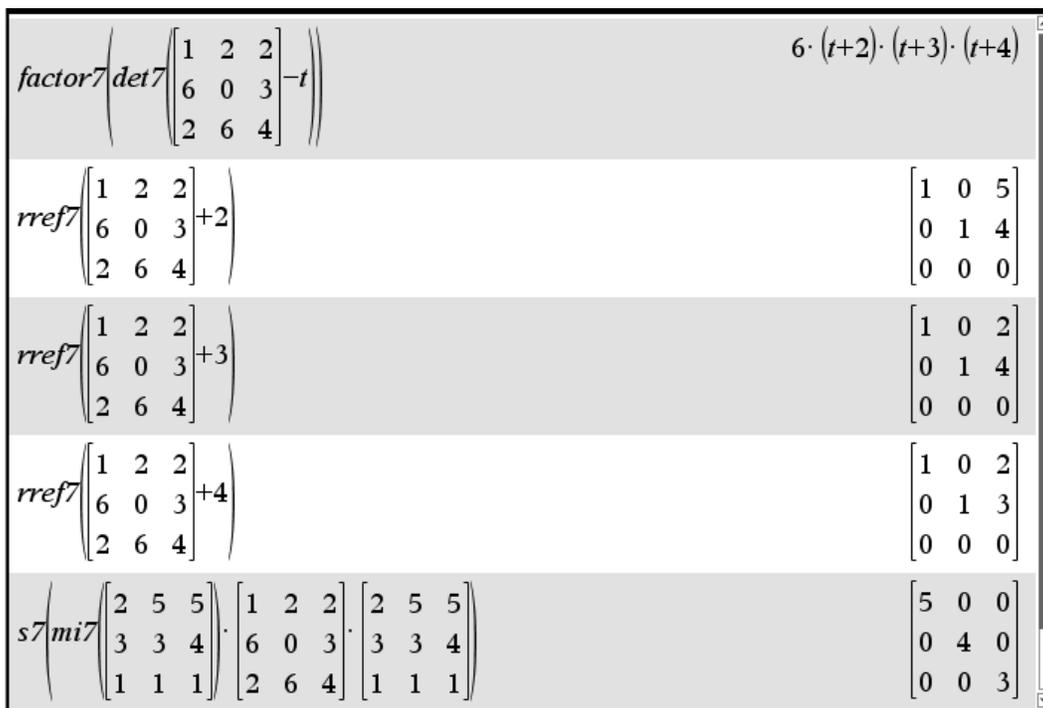
Let's solve $\begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 5 \\ 6 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$ using the inverse matrix $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 4 & 0 & 1 \end{bmatrix}$, and check by row reduction.



3. Let's diagonalize $\begin{bmatrix} 1 & 2 & 2 \\ 6 & 0 & 3 \\ 2 & 6 & 4 \end{bmatrix}$ over \mathbb{F}_7 :

- Find its characteristic polynomial, factor it, find the eigenvalues.
- For each of the eigenvalues, find the eigenvectors.
- Compute $Q^{-1}MQ = D$

We'll let the TI-Nspire do all the work. Note e.g. $-5 = 2$ and $-4 = 3$ in \mathbb{F}_7



• Characteristic polynomial $p(t) = 6(t+2)(t+3)(t+4)$. Eigenvalues $t = 5, 4, 3$.

• Eigenvectors: $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ for $t = 5$, $\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$ for $t = 4$, $\begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}$ for $t = 3$.

• $Q^{-1}MQ = \begin{bmatrix} 2 & 5 & 5 \\ 3 & 3 & 4 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 & 2 \\ 6 & 0 & 3 \\ 2 & 6 & 4 \end{bmatrix} \begin{bmatrix} 2 & 5 & 5 \\ 3 & 3 & 4 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

4. Solve $\begin{cases} ax + y + bz = b \\ bx + ay + bz = b \\ ax + by = 1 \end{cases}$ over \mathbb{F}_4 using **rref4** or a matrix inverse.

Calculator screen showing the solution for problem 4:

$$\text{rref4} \begin{pmatrix} a & 1 & b & b \\ b & a & b & b \\ a & b & 0 & 1 \end{pmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\text{mi4} \begin{pmatrix} a & 1 & b \\ b & a & b \\ a & b & 0 \end{pmatrix} \quad \begin{bmatrix} a & a & a \\ 1 & 1 & b \\ 1 & b & 0 \end{bmatrix}$$

$$\text{s4} \begin{pmatrix} a & a & a \\ 1 & 1 & b \\ 1 & b & 0 \end{pmatrix} \cdot \begin{bmatrix} b \\ b \\ 1 \end{bmatrix} \quad \begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$$

5. Find a basis of $W = \text{span} \left\{ \begin{bmatrix} 1 \\ b \\ 1 \\ a \end{bmatrix}, \begin{bmatrix} b \\ a \\ b \\ 1 \end{bmatrix}, \begin{bmatrix} a \\ b \\ a \\ b \end{bmatrix}, \begin{bmatrix} b \\ b \\ b \\ 1 \end{bmatrix}, \begin{bmatrix} b \\ a \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ b \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ a \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ a \\ a \end{bmatrix} \right\}$

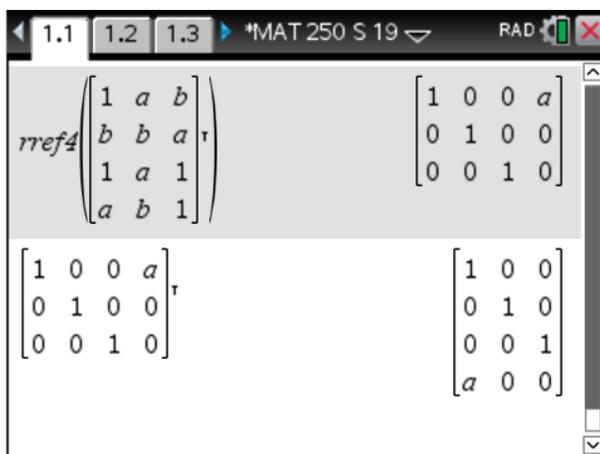
Calculator screen showing the rref4 of an 8x8 matrix:

$$\text{rref4} \begin{pmatrix} 1 & b & a & b & b & 0 & 1 & 1 \\ b & a & b & b & a & b & 1 & 1 \\ 1 & b & a & b & 1 & 1 & 1 & a \\ a & 1 & b & 1 & 1 & 0 & a & a \end{pmatrix}$$

$$\begin{bmatrix} 1 & b & 0 & a & 0 & 1 & b & a \\ 0 & 0 & 1 & b & 0 & a & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & b & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence $\left\{ \begin{bmatrix} 1 \\ b \\ 1 \\ a \end{bmatrix}, \begin{bmatrix} a \\ b \\ a \\ b \end{bmatrix}, \begin{bmatrix} b \\ a \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis of W . In fact, it is not too hard to find a

more “standard” basis for W : $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ a \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$



6. Let's compute the LU decomposition of $\begin{bmatrix} a & a & a & b \\ 0 & 0 & a & b \\ 1 & 0 & 1 & 1 \\ 1 & b & 0 & b \end{bmatrix}$ over \mathbb{F}_4 :

$$\text{lup}_4 \begin{pmatrix} a & a & a & b \\ 0 & 0 & a & b \\ 1 & 0 & 1 & 1 \\ 1 & b & 0 & b \end{pmatrix}$$

With permutation matrix $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ b & a & b & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} a & a & a & b \\ 0 & 1 & 0 & b \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{bmatrix}$$

$P*A = L*U$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} a & a & a & b \\ 0 & 0 & a & b \\ 1 & 0 & 1 & 1 \\ 1 & b & 0 & b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ b & a & b & 1 \end{bmatrix} * \begin{bmatrix} a & a & a & b \\ 0 & 1 & 0 & b \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{bmatrix}$$

Done

etc.