Intersection and Sum of Subspaces

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Outline

- Intersecting Subspaces and Relation Form

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- Summing Subspaces

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- Favorite Problems

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- Favorite Problems
- Complements and Projection Maps

Why?

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- Better understanding of subspaces

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- Better understanding of subspaces
- Gives rise to interesting problems
- Preparation for more advanced STEM fields

Intersection Problem

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$$S_1 = \operatorname{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right), \qquad S_2 = \operatorname{span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}6\\4\\5\end{bmatrix}\right).$$

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- Problem: Find a basis for $S_1 \cap S_2$.

Relation Form

- Change to relation form

$$S_1 = \operatorname{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \left\{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\middle| x_3 = 0\right\},$$

$$S_2 = \operatorname{span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}6\\4\\5\end{bmatrix}\right) = \left\{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\middle| 2x_1 - 13x_2 + 8x_3 = 0\right\}.$$

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- Therefore,

$$S_1 \cap S_2 = \left\{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\middle| 2x_1 - 13x_2 + 8x_3 = 0, x_3 = 0\right\}.$$

Relation Form to Span form

- To find $S_1 \cap S_2$, we solve $2x_1 - 13x_2 + 8x_3 = 0, x_3 = 0$:

$$\left[\begin{array}{ccc|c} 2 & -13 & 8 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -\frac{13}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right],$$

  so x_2 is free, $x_1 = \frac{13}{2}x_2, x_3 = 0$.

- Thus,

$$S_1 \cap S_2 = \left\{ \begin{bmatrix} \frac{13}{2}x_2 \\ x_2 \\ 0 \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} \frac{13}{2} \\ 1 \\ 0 \end{bmatrix} \right).$$

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- Solving a linear system is converting from relation form to span form.

Span form to relation form

- How did we find

$$\operatorname{span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}6\\4\\5\end{bmatrix}\right) = \left\{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \middle| 2x_1 - 13x_2 + 8x_3 = 0\right\}?$$

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- We have $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\in\operatorname{span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}6\\4\\5\end{bmatrix}\right)$ when

$$\left[\begin{array}{cc|c}1&6&x_1\\2&4&x_2\\3&5&x_3\end{array}\right]\sim\left[\begin{array}{cc|c}1&6&x_1\\0&-8&-2x_1+x_2\\0&0&2x_1-13x_2+8x_3\end{array}\right]$$

has a solution, which is when $2x_1-13x_2+8x_3=0$.

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- Easy to test if an element lies in the subspace
- Many subspaces come expressed in relation form
- Disadvantage: Hard to say what the elements of the subspace look like.

Multi-augmented matrices

- Let augmented columns denote coefficients of each variable:

$$
\begin{array}{ccccc} & & x_1 & x_2 & x_3 \end{array}
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\left[\begin{array}{cc|ccc} 1 & 6 & 1 & 0 & 0 \\ 2 & 4 & 0 & 1 & 0 \\ 3 & 5 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|ccc} 1 & 6 & 1 & 0 & 0 \\ 0 & -8 & -2 & 1 & 0 \\ 0 & 0 & 2 & -13 & 8 \end{array}\right]
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- This is a great way to understand finding inverses. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $[T] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \implies \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right],$$

so $[T^{-1}] = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.

Why intersect?

- Question: Suppose $(\vec{u_1}, \ldots, \vec{u_k})$ and $(\vec{v_1}, \ldots, \vec{v_m})$ are linearly independent in $\mathbb{R}^n$. In terms of

$$S_1 = \operatorname{span}(\vec{u_1}, \ldots, \vec{u_k}), \quad S_2 = \operatorname{span}(\vec{v_1}, \ldots, \vec{v_m}),$$

  when is $(\vec{u_1}, \ldots, \vec{u_k}, \vec{v_1}, \ldots, \vec{v_m})$ linearly independent?

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- Answer: $(\vec{u_1}, \ldots, \vec{u_k}, \vec{v_1}, \ldots, \vec{v_m})$ is linearly independent if and only if $S_1 \cap S_2 = \{\vec{0}\}$. (Good proof question for students)

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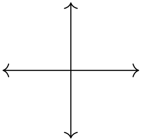
- For matrices A, B, $\ker\left(\begin{bmatrix} A \\ B \end{bmatrix}\right) = \ker(A) \cap \ker(B)$.

Unioning Subspaces

- The union of two subpaces need not be a subspace:

$$\operatorname{span}\left(\begin{bmatrix}1\\0\end{bmatrix}\right) \cup \operatorname{span}\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \left\{ \begin{bmatrix}x_1\\x_2\end{bmatrix} \middle| x_1 = 0, \text{ or } x_2 = 0 \right\}.$$



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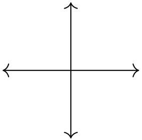

- Want an analog of union for subspaces: sum. For subspaces S_1, S_2, let $S_1 + S_2$ denote the smallest subspace containing S_1 and S_2.

Summing Subspaces

- For subspaces S_1, S_2,

$$S_1 + S_2 = \{\vec{x} + \vec{y} \mid \vec{x} \in S_1, \vec{y} \in S_2\}.$$

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- Summing is easy in span form:

$$\mathrm{span}(\vec{u_1}, \ldots, \vec{u_k}) + \mathrm{span}(\vec{v_1}, \ldots, \vec{v_m}) = \mathrm{span}(\vec{u_1}, \ldots, \vec{u_k}, \vec{v_1}, \ldots, \vec{v_m}).$$

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- $\text{range}(\begin{bmatrix} A & B \end{bmatrix}) = \text{range}(A) + \text{range}(B)$.

Dimension Formula

- For subspaces S_1, S_2,

$$\dim(S_1 + S_2) = \dim(S_1) + \dim(S_2) - \dim(S_1 \cap S_2).$$

- Analog of $|X \cup Y| = |X| + |Y| - |X \cap Y|$.

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- Proof outline:
  (i) Prove $S_1 \cap S_2 = \{\vec{0}\}$ case first.
  (ii) Extending a basis of $S_1 \cap S_2$ to S_1 lets us write

$$S_1 = (S_1 \cap S_2) + V, \quad \text{with} \quad V \cap S_2 = \{0\}.$$

  (iii) Since $V \cap S_2 = \{\vec{0}\}$, we can apply the formula.

Favorite Problems

- Find a linear map $T : \mathbb{R}^3 \to \mathbb{R}$ with

$$\ker(T) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_1 + 4x_2 + 3x_3 = 0 \right\}.$$

- Suppose $(\vec{u_1}, \ldots, \vec{u_k})$ is linearly independent in S_1, $\vec{w_1}, \ldots, \vec{w_k} \in S_2$, and $S_1 \cap S_2 = \{\vec{0}\}$. Show that $(\vec{u_1} + \vec{w_1}, \ldots, \vec{u_k} + \vec{w_k})$ is linearly independent.

- Suppose $S_1 \cap S_2 = \{\vec{0}\}$. Show that every $\vec{v} \in S_1 + S_2$ can be written as $\vec{v} = \vec{x} + \vec{y}$ for some <u>unique</u> $\vec{x} \in S_1$ and $\vec{y} \in S_2$.

Challenge Problem

Suppose subspaces $S_1, \ldots, S_k$ have bases $B_1, \ldots, B_k$, respectively. Show that the following are equivalent:

1. If $\vec{v_1} + \cdots + \vec{v_k} = \vec{0}$ with $\vec{v_1} \in S_1, \ldots, \vec{v_k} \in S_k$, then $\vec{v_1} = 0, \ldots, \vec{v_k} = 0$.
2. Every $\vec{x} \in S_1 + \cdots + S_k$ can be written as $\vec{x} = \vec{v_1} + \cdots + \vec{v_k}$ with $\vec{v_1} \in S_1, \ldots, \vec{v_k} \in S_k$ in a unique way.
3. $B_1 \cup \cdots \cup B_k$ is linearly independent.
4. $\dim(S_1 + \cdots + S_k) = \dim(S_1) + \cdots + \dim(S_k)$.
5. $S_i \cap (S_1 + \cdots + S_{i-1} + S_{i+1} + \cdots + S_k) = \{\vec{0}\}$ for all $i = 1, \ldots, k$.

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2. Every $\vec{x} \in S_1 + \cdots + S_k$ can be written as $\vec{x} = \vec{v_1} + \cdots + \vec{v_k}$ with $\vec{v_1} \in S_1, \dots, \vec{v_k} \in S_k$ in a unique way.
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5. $S_i \cap (S_1 + \cdots + S_{i-1} + S_{i+1} + \cdots + S_k) = \{\vec{0}\}$ for all $i = 1, \dots, k$.

These are properties of eigenspaces of a linear map.

Complementary Subspaces

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- When we transition to subspaces, complements lose uniquess. For example, any two different lines through $\vec{0}$ in $\mathbb{R}^2$ are complementary.

Complement Decomposition and Projections

- If S_1, S_2 are complementary subspaces of $\mathbb{R}^n$, then every $\vec{v} \in \mathbb{R}^n$ can be written as

$$\vec{v} = \vec{x} + \vec{y}$$

for some unique $\vec{x} \in S_1, \vec{y} \in S_2$.

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- We can then define the projection map $\pi_{S_1,S_2} : \mathbb{R}^n \to S_1$ by

$$\pi_{S_1,S_2}(\vec{x} + \vec{y}) = \vec{x} \text{ for all } \vec{x} \in S_1, \vec{y} \in S_2.$$

Properties of Projections

For complementary subspaces S_1, S_2 of $\mathbb{R}^n$, the projection $\pi_{S_1,S_2} : \mathbb{R}^n \to S_1$ satisfies

- π_{S_1,S_2} is linear.
- $\text{range}(\pi_{S_1,S_2}) = S_1$,
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- $\pi_{S_1,S_2}(\vec{x}) = \vec{x}$ for all $\vec{x} \in S_1$.
- $\pi^2_{S_1,S_2} = \pi_{S_1,S_2}$.

Favorite Problems

- If $S_1, S_2 \subseteq \mathbb{R}^n$ are complementary subspaces, and $T_1 : S_1 \to \mathbb{R}^m$, $T_2 : S_2 \to \mathbb{R}^m$ are linear maps, then there exists a unique linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ so that

$$T\mid_{S_1} = T_1,\, T\mid_{S_2} = T_2.$$

- If $P : \mathbb{R}^n \to \mathbb{R}^n$ is linear with $P^2 = P$, then P is a projection map.
- For any subspace S and complementary subspaces S_1, S_2,

$$\pi_{S_1,S_2}(S) = (S + S_2) \cap S_1,$$
$$\pi^{-1}_{S_1,S_2}(S) = (S \cap S_1) + S_2.$$

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- If you only have time to include one new idea, introduce relation form and converting between relation and span form.
- For more details and problems, see https://sites.math.washington.edu/∼ahlbach/linaltextbook/
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