Powers of Matrices and Exponential Matrices

Jeffrey Yeh, Chon In (Dave) Luk, Lyheng Phey, John Kath, Tanner Thomas and Luis Cervantes

All Authors are Cal Poly Pomona Students or Alumni

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Outline

Assumptions:

Suppose M is a real $n \times n$ matrix having n distinct exact eigenvalues $\lambda_0, \lambda_1, ..., \lambda_{n-1}$

Objectives:

Find exact expressions for

- M^k, $k \in \mathbb{N}$
- $\exp(Mt)$, where t is time

Applications:

- $M = P$: one-step probability transition matrix of Markov chain
- $M = Q$: rate matrix of a Markov process

Diagonalization Method: $M = S \cdot D \cdot S^{-1}$

For a real $n \times n$ matrix M that has n distinct eigenvalues $\lambda_0, \lambda_1, ..., \lambda_{n-1}$,

$$M^k = S \cdot D^k \cdot S^{-1} \qquad \text{for any } k \in \mathbb{N}$$

$$e^{Mt} = S \cdot e^{Dt} \cdot S^{-1} \qquad \text{where } t \text{ is continuous time}$$

$$D = \begin{bmatrix} \lambda_0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1} \end{bmatrix} \text{ and } e^{Dt} = \begin{bmatrix} e^{\lambda_0 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{(\lambda_{n-1})t} \end{bmatrix}$$

S is a $n \times n$ matrix where each of the columns of S is an eigenvector corresponding to each distinct eigenvalues.

Cayley-Hamilton/Linear Recurrence Relation Method

For a real $n \times n$ matrix M that has n distinct eigenvalues $\lambda_0, \lambda_1, ..., \lambda_{n-1}$,

$$M^k = A_0\lambda_0^k + A_1\lambda_1^k + A_2\lambda_2^k + ... + A_{n-1}\lambda_{n-1}^k$$

for any $k \in \mathbb{N}$

$$e^{Mt} = A_0 e^{\lambda_0 t} + A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + ... + A_{n-1} e^{(\lambda_{n-1})t}$$

where t is continuous time and

where $A_0, A_1, ..., A_{n-1}$ are constant $n \times n$ matrices.

Diagonalization of 2×2 Matrix: $M^k = SD^kS^{-1} \quad k \in \mathbb{N}$

Consider this 2×2 matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Assume

$$\lambda_0 = \frac{a+d-\sqrt{(a-d)^2+4bc}}{2} \quad \text{and} \quad \lambda_1 = \frac{a+d+\sqrt{(a-d)^2+4bc}}{2}$$

are the distinct eigenvalues, then

$$S = \begin{bmatrix} \frac{a-d-\sqrt{(a-d)^2+4bc}}{2c} & \frac{a-d+\sqrt{(a-d)^2+4bc}}{2c} \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} \frac{a+d-\sqrt{(a-d)^2+4bc}}{2} & 0 \\ 0 & \frac{a+d+\sqrt{(a-d)^2+4bc}}{2} \end{bmatrix}$$

Cayley-Hamilton 2×2 Matrix: $M^k = A_0\lambda_0^k + A_1\lambda_1^k \quad k \in \mathbb{N}$

For the same 2×2 matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$M^k = A_0\lambda_0^k + A_1\lambda_1^k = \left(-\frac{1}{\sqrt{(a-d)^2 + 4bc}} \begin{bmatrix} a - \lambda_1 & b \\ c & d - \lambda_1 \end{bmatrix} \right) \lambda_0^k$$

$$+ \left(\frac{1}{\sqrt{(a-d)^2 + 4bc}} \begin{bmatrix} a - \lambda_0 & b \\ c & d - \lambda_0 \end{bmatrix} \right) \lambda_1^k$$

where the distinct eigenvalues are

$$\lambda_0 = \frac{a + d - \sqrt{(a-d)^2 + 4bc}}{2} \quad \text{and} \quad \lambda_1 = \frac{a + d + \sqrt{(a-d)^2 + 4bc}}{2}$$

Vandermonde Matrix & Inverse

From Cayley-Hamilton/Linear Recurrence Relation Approach, we want to solve the following system of linear equations for A's.

$$\begin{aligned}
\mathrm{I} &= A_0 + A_1 + \cdots + A_{n-1} \\
P^1 &= A_0\lambda_0 + A_1\lambda_1 + \cdots + A_{n-1}\lambda_{n-1} \\
P^2 &= A_0\lambda_0^2 + A_1\lambda_1^2 + \cdots + A_{n-1}\lambda_{n-1}^2 \\
&\vdots \\
P^{n-1} &= A_0\lambda_0^{n-1} + A_1\lambda_1^{n-1} + \cdots + A_{n-1}\lambda_{n-1}^{n-1}
\end{aligned}$$

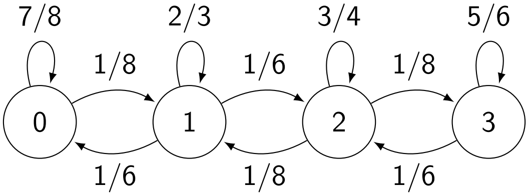
$$\underbrace{\begin{bmatrix} \mathrm{I} \\ P^1 \\ P^2 \\ \vdots \\ P^{n-1} \end{bmatrix}}_{\mathcal{P}} = \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_0 & \lambda_1 & \cdots & \lambda_{n-1} \\ \lambda_0^2 & \lambda_1^2 & \cdots & \lambda_{n-1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_0^{n-1} & \lambda_1^{n-1} & \cdots & \lambda_{n-1}^{n-1} \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_{n-1} \end{bmatrix}}_{\mathcal{A}}$$

$$\mathcal{P} = V\mathcal{A}, \text{ then } \mathcal{A} = V^{-1}\mathcal{P}$$

In order to solve for A's, we have to find the inverse of V, where V is a Vandermonde Matrix. Each entry of the V^{-1} can be computed explicitly.

Example 1

Consider the following birth-death chain:



$$P = \begin{bmatrix} \frac{7}{8} & \frac{1}{8} & 0 & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 \\ 0 & \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \\ 0 & 0 & \frac{1}{6} & \frac{5}{6} \end{bmatrix}$$

The eigenvalues of this birth-death chain are:

$$\lambda_0 = \frac{1}{2}, \quad \lambda_1 = \frac{11}{12}$$

$$\lambda_2 = \frac{17}{24}, \quad \lambda_3 = 1$$

Diagonalization Approach: $P = S \cdot D \cdot S^{-1}$

$$P^k = \begin{bmatrix} \frac{7}{8} & \frac{1}{8} & 0 & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 \\ 0 & \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \\ 0 & 0 & \frac{1}{6} & \frac{5}{6} \end{bmatrix}^k = S \cdot D^k \cdot S^{-1}$$

$$S = \begin{bmatrix} -1 & -1 & \frac{9}{16} & 1 \\ 3 & -\frac{1}{3} & -\frac{3}{4} & 1 \\ -2 & \frac{1}{2} & -\frac{3}{4} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \; D^k = \begin{bmatrix} \left(\frac{1}{2}\right)^k & 0 & 0 & 0 \\ 0 & \left(\frac{11}{12}\right)^k & 0 & 0 \\ 0 & 0 & \left(\frac{17}{24}\right)^k & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; S^{-1} = \begin{bmatrix} -\frac{2}{25} & \frac{9}{50} & -\frac{4}{25} & \frac{3}{50} \\ -\frac{12}{25} & -\frac{3}{25} & \frac{6}{25} & \frac{9}{25} \\ \frac{48}{175} & -\frac{48}{175} & -\frac{64}{175} & \frac{64}{175} \\ \frac{2}{7} & \frac{3}{14} & \frac{2}{7} & \frac{3}{14} \end{bmatrix}$$

Cayley-Hamilton/Linear Recurrence Relation Approach

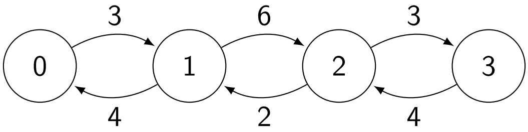
Using Dave's Python computer program, we have: $P^k = A_0\lambda_0^k + A_1\lambda_1^k + A_2\lambda_2^k + A_3$

$$P^k = \begin{bmatrix} \frac{2}{25} & \frac{-9}{50} & \frac{4}{25} & \frac{-3}{50} \\ \frac{-6}{25} & \frac{27}{50} & \frac{-12}{25} & \frac{9}{50} \\ \frac{4}{25} & \frac{-9}{25} & \frac{8}{25} & \frac{-3}{25} \\ \frac{-2}{25} & \frac{9}{50} & \frac{-4}{25} & \frac{3}{50} \end{bmatrix} \left(\frac{1}{2}\right)^k + \begin{bmatrix} \frac{12}{25} & \frac{3}{25} & \frac{-6}{25} & \frac{-9}{25} \\ \frac{4}{25} & \frac{1}{25} & \frac{-2}{25} & \frac{-3}{25} \\ \frac{-6}{25} & \frac{-3}{50} & \frac{3}{25} & \frac{9}{50} \\ \frac{-12}{25} & \frac{-3}{25} & \frac{6}{25} & \frac{9}{25} \end{bmatrix} \left(\frac{11}{12}\right)^k$$

$$+ \begin{bmatrix} \frac{27}{175} & \frac{-27}{175} & \frac{-36}{175} & \frac{36}{175} \\ \frac{-36}{175} & \frac{36}{175} & \frac{48}{175} & \frac{-48}{175} \\ \frac{-36}{175} & \frac{36}{175} & \frac{48}{175} & \frac{-48}{175} \\ \frac{48}{175} & \frac{-48}{175} & \frac{-64}{175} & \frac{64}{175} \end{bmatrix} \left(\frac{17}{24}\right)^k + \begin{bmatrix} \frac{2}{7} & \frac{3}{14} & \frac{2}{7} & \frac{3}{14} \\ \frac{2}{7} & \frac{3}{14} & \frac{2}{7} & \frac{3}{14} \\ \frac{2}{7} & \frac{3}{14} & \frac{2}{7} & \frac{3}{14} \\ \frac{2}{7} & \frac{3}{14} & \frac{2}{7} & \frac{3}{14} \end{bmatrix}$$

Example 2

Consider the following diagram which represents a birth-death process:



We wish to determine $P(t) = [P_{ij}(t)]\ i, j \in \{0, 1, 2, 3\}$. It is known that:

$$P(t) = e^{Qt} \text{ where } Q = \begin{bmatrix} -3 & 3 & 0 & 0 \\ 4 & -10 & 6 & 0 \\ 0 & 2 & -5 & 3 \\ 0 & 0 & 4 & -4 \end{bmatrix}$$

We can find exact eigenvalues of Q by Professor Kouachi [4]:

$$\lambda_0 = -13, \quad \lambda_1 = -2, \quad \lambda_2 = -7, \quad \lambda_3 = 0.$$

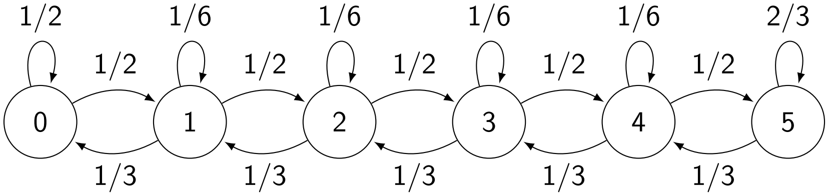
Using Luis Cervantes' MATLAB program, the solution is given by:

$$P(t) = \begin{bmatrix} \frac{36}{55} & \frac{9}{55} & \frac{-18}{55} & \frac{-27}{55} \\ \frac{12}{55} & \frac{3}{55} & \frac{-6}{55} & \frac{-9}{55} \\ \frac{-8}{55} & \frac{-2}{55} & \frac{4}{55} & \frac{6}{55} \\ \frac{-16}{55} & \frac{-4}{55} & \frac{-8}{55} & \frac{12}{55} \end{bmatrix} e^{-2t} + \begin{bmatrix} \frac{3}{35} & \frac{-3}{35} & \frac{-9}{35} & \frac{9}{35} \\ \frac{-4}{35} & \frac{4}{35} & \frac{12}{35} & \frac{-12}{35} \\ \frac{-4}{35} & \frac{4}{35} & \frac{12}{35} & \frac{-12}{35} \\ \frac{16}{105} & \frac{-16}{105} & \frac{-16}{35} & \frac{16}{35} \end{bmatrix} e^{-7t}$$

$$+ \begin{bmatrix} \frac{12}{143} & \frac{-30}{143} & \frac{27}{143} & \frac{-9}{143} \\ \frac{-40}{143} & \frac{100}{143} & \frac{-90}{143} & \frac{30}{143} \\ \frac{12}{143} & \frac{-30}{143} & \frac{27}{143} & \frac{-9}{143} \\ \frac{-16}{429} & \frac{40}{429} & \frac{-12}{143} & \frac{4}{143} \end{bmatrix} e^{-13t} + \begin{bmatrix} \frac{16}{91} & \frac{12}{91} & \frac{36}{91} & \frac{27}{91} \\ \frac{16}{91} & \frac{12}{91} & \frac{36}{91} & \frac{27}{91} \\ \frac{16}{91} & \frac{12}{91} & \frac{36}{91} & \frac{27}{91} \\ \frac{16}{91} & \frac{12}{91} & \frac{36}{91} & \frac{27}{91} \end{bmatrix} e^{0}$$

Example 3

Consider the following birth-death chain:



$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

The eigenvalues of this birth-death chain are given by Kouachi [4]:

$\lambda_0 = \frac{1}{6}\left(1 - 3\sqrt{2}\right)$, $\lambda_1 = \frac{1}{6}\left(1 - \sqrt{6}\right)$,

$\lambda_2 = \frac{1}{6}\left(1 + 3\sqrt{2}\right)$, $\lambda_3 = \frac{1}{6}\left(1 + \sqrt{6}\right)$,

$\lambda_4 = \frac{1}{6}$, $\lambda_5 = 1$

Using Dave's computer program, we have:

$$P^k = A_0\lambda_0^k + A_1\lambda_1^k + A_2\lambda_2^k + A_3\lambda_3^k + A_4\lambda_4^k + A_5$$

$$A_0 = \begin{bmatrix}
-\frac{3\sqrt{2}}{28}+\frac{5}{28} & -\frac{9\sqrt{2}}{56}+\frac{1}{7} & -\frac{3\sqrt{2}}{56}+\frac{3}{14} & -\frac{9\sqrt{2}}{112}-\frac{3}{56} & -\frac{9}{112}+\frac{9\sqrt{2}}{56} & -\frac{45}{112}+\frac{27\sqrt{2}}{112} \\
-\frac{3\sqrt{2}}{28}+\frac{2}{21} & -\frac{\sqrt{2}}{28}+\frac{19}{84} & -\frac{5\sqrt{2}}{28}-\frac{1}{28} & \frac{3\sqrt{2}}{28}+\frac{11}{56} & -\frac{15}{56}-\frac{3\sqrt{2}}{112} & -\frac{3}{14}+\frac{27\sqrt{2}}{112} \\
-\frac{\sqrt{2}}{42}+\frac{2}{21} & -\frac{5\sqrt{2}}{42}-\frac{1}{42} & \frac{\sqrt{2}}{14}+\frac{3}{14} & -\frac{\sqrt{2}}{7}-\frac{5}{28} & \frac{3}{28}+\frac{9\sqrt{2}}{56} & -\frac{3}{14}+\frac{3\sqrt{2}}{56} \\
-\frac{\sqrt{2}}{42}-\frac{1}{63} & \frac{\sqrt{2}}{21}+\frac{11}{126} & -\frac{2\sqrt{2}}{21}-\frac{5}{42} & \frac{3\sqrt{2}}{28}+\frac{13}{84} & -\frac{1}{7}-\frac{5\sqrt{2}}{56} & \frac{1}{28}+\frac{3\sqrt{2}}{56} \\
-\frac{1}{63}+\frac{2\sqrt{2}}{63} & -\frac{5}{63}-\frac{\sqrt{2}}{126} & \frac{1}{21}+\frac{\sqrt{2}}{14} & -\frac{2}{21}-\frac{5\sqrt{2}}{84} & \frac{\sqrt{2}}{28}+\frac{3}{28} & -\frac{\sqrt{2}}{14}+\frac{1}{28} \\
-\frac{10}{189}+\frac{2\sqrt{2}}{63} & -\frac{8}{189}+\frac{\sqrt{2}}{21} & -\frac{4}{63}+\frac{\sqrt{2}}{63} & \frac{1}{63}+\frac{\sqrt{2}}{42} & -\frac{\sqrt{2}}{21}+\frac{1}{42} & -\frac{\sqrt{2}}{14}+\frac{5}{42}
\end{bmatrix}$$

$$A_1 = \begin{bmatrix}
-\frac{3\sqrt{6}}{76}+\frac{15}{76} & -\frac{9\sqrt{6}}{152}-\frac{3}{38} & -\frac{9}{76}+\frac{15\sqrt{6}}{152} & -\frac{27}{152}+\frac{45\sqrt{6}}{304} & -\frac{81}{304}-\frac{9\sqrt{6}}{152} & -\frac{27\sqrt{6}}{304}+\frac{135}{304} \\
-\frac{3\sqrt{6}}{76}-\frac{1}{19} & \frac{5\sqrt{6}}{76}+\frac{13}{76} & -\frac{9}{76}-\frac{\sqrt{6}}{38} & -\frac{27}{152}-\frac{3\sqrt{6}}{76} & \frac{45}{152}+\frac{39\sqrt{6}}{304} & -\frac{27\sqrt{6}}{304}-\frac{9}{76} \\
-\frac{1}{19}+\frac{5\sqrt{6}}{114} & -\frac{3}{38}-\frac{\sqrt{6}}{57} & -\frac{\sqrt{6}}{38}+\frac{5}{38} & -\frac{3\sqrt{6}}{76}+\frac{15}{76} & -\frac{9\sqrt{6}}{152}-\frac{3}{38} & -\frac{9}{76}+\frac{15\sqrt{6}}{152} \\
-\frac{1}{19}+\frac{5\sqrt{6}}{114} & -\frac{3}{38}-\frac{\sqrt{6}}{57} & -\frac{\sqrt{6}}{38}+\frac{5}{38} & -\frac{3\sqrt{6}}{76}+\frac{15}{76} & -\frac{9\sqrt{6}}{152}-\frac{3}{38} & -\frac{9}{76}+\frac{15\sqrt{6}}{152} \\
-\frac{1}{19}-\frac{2\sqrt{6}}{171} & \frac{5}{57}+\frac{13\sqrt{6}}{342} & -\frac{\sqrt{6}}{38}-\frac{2}{57} & -\frac{3\sqrt{6}}{76}-\frac{1}{19} & \frac{5\sqrt{6}}{76}+\frac{13}{76} & -\frac{9}{76}-\frac{\sqrt{6}}{38} \\
-\frac{2\sqrt{6}}{171}+\frac{10}{171} & -\frac{\sqrt{6}}{57}-\frac{4}{171} & -\frac{2}{57}+\frac{5\sqrt{6}}{171} & -\frac{1}{19}+\frac{5\sqrt{6}}{114} & -\frac{3}{38}-\frac{\sqrt{6}}{57} & -\frac{\sqrt{6}}{38}+\frac{5}{38}
\end{bmatrix}$$

$$A_2 = \begin{bmatrix} \frac{3\sqrt{2}}{28}+\frac{5}{28} & \frac{1}{7}+\frac{9\sqrt{2}}{56} & \frac{3\sqrt{2}}{56}+\frac{3}{14} & -\frac{3}{56}+\frac{9\sqrt{2}}{112} & -\frac{9\sqrt{2}}{56}-\frac{9}{112} & -\frac{45}{112}-\frac{27\sqrt{2}}{112} \\ \frac{2}{21}+\frac{3\sqrt{2}}{28} & \frac{\sqrt{2}}{28}+\frac{19}{84} & -\frac{1}{28}+\frac{5\sqrt{2}}{28} & -\frac{3\sqrt{2}}{28}+\frac{11}{56} & -\frac{15}{56}+\frac{3\sqrt{2}}{112} & -\frac{27\sqrt{2}}{112}-\frac{3}{14} \\ \frac{\sqrt{2}}{42}+\frac{2}{21} & -\frac{1}{42}+\frac{5\sqrt{2}}{42} & -\frac{\sqrt{2}}{14}+\frac{3}{14} & -\frac{5}{28}+\frac{\sqrt{2}}{7} & -\frac{9\sqrt{2}}{56}+\frac{3}{28} & -\frac{3}{14}-\frac{3\sqrt{2}}{56} \\ -\frac{1}{63}+\frac{\sqrt{2}}{42} & -\frac{\sqrt{2}}{21}+\frac{11}{126} & -\frac{5}{42}+\frac{2\sqrt{2}}{21} & -\frac{3\sqrt{2}}{28}+\frac{13}{84} & -\frac{1}{7}+\frac{5\sqrt{2}}{56} & -\frac{3\sqrt{2}}{56}+\frac{1}{28} \\ -\frac{2\sqrt{2}}{63}-\frac{1}{63} & -\frac{5}{63}+\frac{\sqrt{2}}{126} & -\frac{\sqrt{2}}{14}+\frac{1}{21} & -\frac{2}{21}+\frac{5\sqrt{2}}{84} & -\frac{\sqrt{2}}{28}+\frac{3}{28} & \frac{1}{28}+\frac{\sqrt{2}}{14} \\ -\frac{10}{189}-\frac{2\sqrt{2}}{63} & -\frac{\sqrt{2}}{21}-\frac{8}{189} & -\frac{4}{63}-\frac{\sqrt{2}}{63} & -\frac{\sqrt{2}}{42}+\frac{1}{63} & \frac{1}{42}+\frac{\sqrt{2}}{21} & \frac{\sqrt{2}}{14}+\frac{5}{42} \end{bmatrix}$$

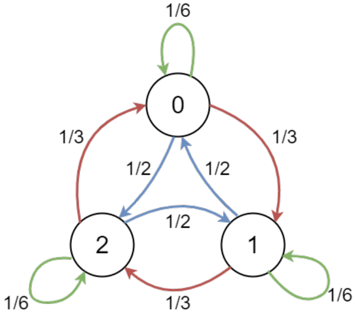
$$A_3 = \begin{bmatrix} \frac{3\sqrt{6}}{76}+\frac{15}{76} & -\frac{3}{38}+\frac{9\sqrt{6}}{152} & -\frac{15\sqrt{6}}{152}-\frac{9}{76} & -\frac{45\sqrt{6}}{304}-\frac{27}{152} & -\frac{81}{304}+\frac{9\sqrt{6}}{152} & \frac{27\sqrt{6}}{304}+\frac{135}{304} \\ -\frac{1}{19}+\frac{3\sqrt{6}}{76} & -\frac{5\sqrt{6}}{76}+\frac{13}{76} & -\frac{9}{76}+\frac{\sqrt{6}}{38} & -\frac{27}{152}+\frac{3\sqrt{6}}{76} & -\frac{39\sqrt{6}}{304}+\frac{45}{152} & -\frac{9}{76}+\frac{27\sqrt{6}}{304} \\ -\frac{5\sqrt{6}}{114}-\frac{1}{19} & -\frac{3}{38}+\frac{\sqrt{6}}{57} & \frac{\sqrt{6}}{38}+\frac{5}{38} & \frac{3\sqrt{6}}{76}+\frac{15}{76} & -\frac{3}{38}+\frac{9\sqrt{6}}{152} & -\frac{15\sqrt{6}}{152}-\frac{9}{76} \\ -\frac{5\sqrt{6}}{114}-\frac{1}{19} & -\frac{3}{38}+\frac{\sqrt{6}}{57} & \frac{\sqrt{6}}{38}+\frac{5}{38} & \frac{3\sqrt{6}}{76}+\frac{15}{76} & -\frac{3}{38}+\frac{9\sqrt{6}}{152} & -\frac{15\sqrt{6}}{152}-\frac{9}{76} \\ -\frac{1}{19}+\frac{2\sqrt{6}}{171} & -\frac{13\sqrt{6}}{342}+\frac{5}{57} & -\frac{2}{57}+\frac{\sqrt{6}}{38} & -\frac{1}{19}+\frac{3\sqrt{6}}{76} & -\frac{5\sqrt{6}}{76}+\frac{13}{76} & -\frac{9}{76}+\frac{\sqrt{6}}{38} \\ \frac{2\sqrt{6}}{171}+\frac{10}{171} & -\frac{4}{171}+\frac{\sqrt{6}}{57} & -\frac{5\sqrt{6}}{171}-\frac{2}{57} & -\frac{5\sqrt{6}}{114}-\frac{1}{19} & -\frac{3}{38}+\frac{\sqrt{6}}{57} & \frac{\sqrt{6}}{38}+\frac{5}{38} \end{bmatrix}$$

$$A_4 = \begin{bmatrix} \frac{1}{5} & -\frac{1}{5} & -\frac{3}{10} & \frac{3}{10} & \frac{9}{20} & -\frac{9}{20} \\ -\frac{2}{15} & \frac{2}{15} & \frac{1}{5} & -\frac{1}{5} & -\frac{3}{10} & \frac{3}{10} \\ -\frac{2}{15} & \frac{2}{15} & \frac{1}{5} & -\frac{1}{5} & -\frac{3}{10} & \frac{3}{10} \\ \frac{4}{45} & -\frac{4}{45} & -\frac{2}{15} & \frac{2}{15} & \frac{1}{5} & -\frac{1}{5} \\ \frac{4}{45} & -\frac{4}{45} & -\frac{2}{15} & \frac{2}{15} & \frac{1}{5} & -\frac{1}{5} \\ -\frac{8}{135} & \frac{8}{135} & \frac{4}{45} & -\frac{4}{45} & -\frac{2}{15} & \frac{2}{15} \end{bmatrix}$$

$$A_5 = \begin{bmatrix} \frac{32}{665} & \frac{48}{665} & \frac{72}{665} & \frac{108}{665} & \frac{162}{665} & \frac{243}{665} \\ \frac{32}{665} & \frac{48}{665} & \frac{72}{665} & \frac{108}{665} & \frac{162}{665} & \frac{243}{665} \\ \frac{32}{665} & \frac{48}{665} & \frac{72}{665} & \frac{108}{665} & \frac{162}{665} & \frac{243}{665} \\ \frac{32}{665} & \frac{48}{665} & \frac{72}{665} & \frac{108}{665} & \frac{162}{665} & \frac{243}{665} \\ \frac{32}{665} & \frac{48}{665} & \frac{72}{665} & \frac{108}{665} & \frac{162}{665} & \frac{243}{665} \\ \frac{32}{665} & \frac{48}{665} & \frac{72}{665} & \frac{108}{665} & \frac{162}{665} & \frac{243}{665} \end{bmatrix}$$

Example 4

Consider the following Circular Markov Chain:



$$P = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

P is a 3×3 **circulant** matrix

Eigenvalues are given by Davis [8]:

$$\lambda_1 = -\frac{1}{4} - \frac{\sqrt{3}i}{12},\ \lambda_2 = -\frac{1}{4} + \frac{\sqrt{3}i}{12}$$

$$\lambda_0 = 1$$

Using Dave's computer program, we have : $P^k = A_1\lambda_1^k + A_2\lambda_2^k + A_0$

$$\begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{bmatrix}^k = \begin{bmatrix} \frac{1}{3} & -\frac{1}{6}-\frac{\sqrt{3}i}{6} & -\frac{1}{6}+\frac{\sqrt{3}i}{6} \\ -\frac{1}{6}+\frac{\sqrt{3}i}{6} & \frac{1}{3} & -\frac{1}{6}-\frac{\sqrt{3}i}{6} \\ -\frac{1}{6}-\frac{\sqrt{3}i}{6} & -\frac{1}{6}+\frac{\sqrt{3}i}{6} & \frac{1}{3} \end{bmatrix} \left(-\frac{1}{4}-\frac{\sqrt{3}i}{12}\right)^k$$

$$+ \begin{bmatrix} \frac{1}{3} & -\frac{1}{6}+\frac{\sqrt{3}i}{6} & -\frac{1}{6}-\frac{\sqrt{3}i}{6} \\ -\frac{1}{6}-\frac{\sqrt{3}i}{6} & \frac{1}{3} & -\frac{1}{6}+\frac{\sqrt{3}i}{6} \\ -\frac{1}{6}+\frac{\sqrt{3}i}{6} & -\frac{1}{6}-\frac{\sqrt{3}i}{6} & \frac{1}{3} \end{bmatrix} \left(-\frac{1}{4}+\frac{\sqrt{3}i}{12}\right)^k + \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Summary

1. If we have n distinct eigenvalues of a real $n \times n$ matrix M, then we have presented two approaches:
   (a) Diagonalization
   (b) Cayley-Hamilton/Linear Recurrence Relation

   to find the exact formula for M^k and $exp(Mt)$. We have shown examples of different dimensions using computer programs.

2. For the Cayley-Hamilton/Linear Recurrence Relation Approach, using the inverse Vandermonde matrix can accelerate the computation.

3. We are able to find exact eigenvalues for the following classes of matrices with large n:
   (a) Certain tridiagonal matrices studied by Kouachi [4]
   (b) Circulant matrices [8]

References

[1] Lilinoe Harbottle, Black Hunter and Alan Krinik, *The Irreducible Three and Four-State Markov process*, Integration, Mathematical Theory and Applications, Volume 2, Issue 3, pages 335-355, 2011.

[2] Donald Knuth, *The Art of Computer Programming Vol 1: Fundamental Algorithms Third Edition*, Addison-Wesley 1997.

[3] Said Kouachi, *Eigenvalues and Eigenvectors of Tridiagonal Matrices*, Electronic Journal of Linear Algebra, Vol. 15,(2006), pages 115-133.

[4] Said Kouachi, *Eigenvalues and Eigenvectors of Some Tridiagonal Matrices with Non-Constant Diagonal Entries*, Applicationes Mathematicae 35, Vol. 1, (2008), pages 107-120.

[5] Alan Krinik and Jeniffer Switkes, *An Element in the kth Power of an $n \times n$ Matrix*, Preprint.

[6] Clever Moler and Charles Van Loan, *Nineteen Dubious Ways to Compute the Exponential of a Matrix, Twenty-Five Years Later*, SIAM REVIEW Vol 45, NO.1, pp. 1-46, 2003.

[7] Ben Noble and James W. Daniel, *Applied Linear Algebra*, Prentice-Hall, New Jersey 1988.

[8] Philip J. Davis, *Circulant Matrices*, Wiley, New York, 1970

If you wish to hear more on this topic, you are invited to AMS Special Session on Markov Chains, Markov Processes and Applications on Friday January 12, 2018, 1:00 p.m. - 6:50 p.m.

Location: Room 29D, Upper Level, San Diego Convention Center
Title: Matrix Properties of a Class of Birth-Death Chains and Processes
Time: 2:00 p.m. - 2:20 p.m.

Thank you!