From Linear Algebra to Cech Cohomology in One Undergraduate Semester

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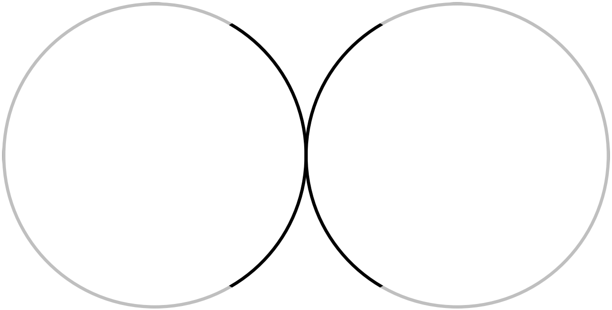
Why am I giving this talk?

- Studying for graduate school oral exams
- First undergraduate research students
- Given two sections of Linear Algebra to teach
- Using Strang's book . . . incidence matrices!
- Personal biases
- Success via anecdotal evidence
- Topological Data Analysis
- This MAA Contributed Paper Session's title!

Presentation Outline

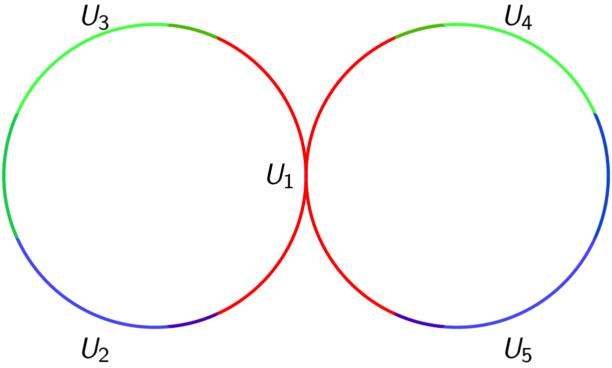
Computing $H^\bullet(S^1 \wedge S^1, \mathbb{R})$: a good open cover

Use open sets of $\mathbb{R}^2$ to *cover* S^1.



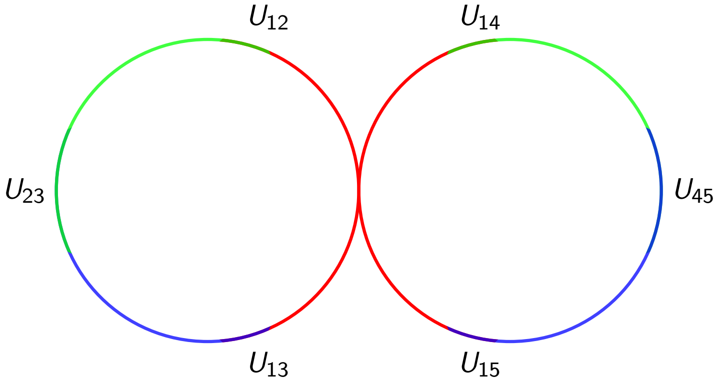
Computing $H^\bullet(S^1 \wedge S^1, \mathbb{R})$: a good open cover

Use open sets of $\mathbb{R}^2$ to *cover* S^1 and call the intersections of these open sets with S^1, U_i.



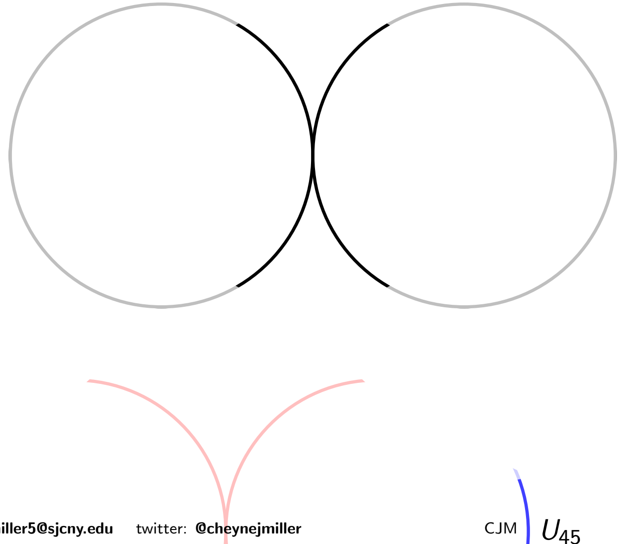
Computing $H^\bullet(S^1 \wedge S^1, \mathbb{R})$: a good open cover

Label the intersections of the U_i's, $U_{ij} := U_i \cap U_j$.



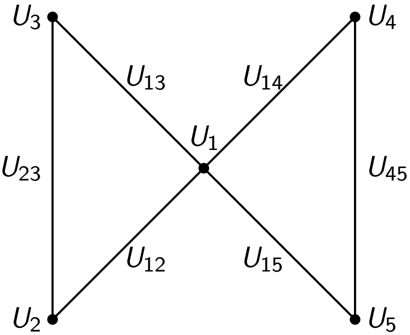
Computing $H^\bullet(S^1 \wedge S^1, \mathbb{R})$: a good open cover

Check that each U_i and each $U_i \cap U_j$ is contractible.



Computing $H^\bullet(S^1 \wedge S^1, \mathbb{R})$: the Cech nerve

Convert your good open cover to its *Cech nerve*.



Computing $H^\bullet(S^1 \wedge S^1, \mathbb{R})$: the Cech complex

$$0 \to C^0 \xrightarrow{\delta^0} C^1 \to 0,$$

where C^0 is generated by the vertices (open sets, U_i) and C^1 is generated by the edges (intersections, U_{ij}).

$$0 \longrightarrow C^0 \cong \mathbb{R}^5 \xrightarrow{\quad \delta^0 \quad} C^1 \cong \mathbb{R}^6 \longrightarrow 0$$

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} \xmapsto{\quad \delta^0 \quad} \begin{pmatrix} f_2 - f_1 \\ f_3 - f_1 \\ f_4 - f_1 \\ f_5 - f_1 \\ f_3 - f_2 \\ f_5 - f_4 \end{pmatrix}$$

- Each $\overline{f} \in C^0$ has components $f_0, f_1, f_2, f_3, f_4, f_5$ and each

Computing $H^\bullet(S^1 \wedge S^1, \mathbb{R})$: the matrix, δ^0

$$0 \to C^0 \xrightarrow{\delta^0} C^1 \to 0 \qquad \delta(\overline{f})_{ij} := f_j - f_i$$

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} = \begin{pmatrix} f_2 - f_1 \\ f_3 - f_1 \\ f_4 - f_1 \\ f_5 - f_1 \\ f_3 - f_2 \\ f_5 - f_4 \end{pmatrix}$$

Computing $H^\bullet(S^1 \wedge S^1, \mathbb{R})$: the image of δ^0

$$0 \to C^0 \xrightarrow{\delta^0} C^1 \to 0 \qquad \delta(\overline{f})_{ij} := f_j - f_i$$

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} = \begin{pmatrix} f_2 - f_1 \\ f_3 - f_1 \\ f_4 - f_1 \\ f_5 - f_1 \\ f_3 - f_2 \\ f_5 - f_4 \end{pmatrix}$$

$$rank(\delta^0) = 4 \quad \Rightarrow \quad im(\delta^0) \cong \mathbb{R}^4$$

Computing $H^\bullet(S^1 \wedge S^1, \mathbb{R})$: the kernel of δ^0

$$0 \to C^0 \xrightarrow{\delta^0} C^1 \to 0 \qquad \delta(\overline{f})_{ij} := f_j - f_i$$

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} = \begin{pmatrix} f_2 - f_1 \\ f_3 - f_1 \\ f_4 - f_1 \\ f_5 - f_1 \\ f_3 - f_2 \\ f_5 - f_4 \end{pmatrix}$$

$$rank(\delta^0) = 4 \quad \Rightarrow \quad ker(\delta^0) \cong \mathbb{R}^{dim(C^0) - rank(\delta^0)} = \mathbb{R}^1$$

A formal definition: the cohomology groups

Definition

Given a cochain complex, i.e. a sequence of vector spaces with linear maps connecting them,

$$\dots \xrightarrow{\delta^{p-2}} C^{p-1} \xrightarrow{\delta^{p-1}} C^{p} \xrightarrow{\delta^{p}} C^{p+1} \xrightarrow{\delta^{p+1}} \dots,$$

where $\delta \circ \delta = 0$, we define the p-th cohomology group, H^p by

$$H^p := ker(\delta^p)/im(\delta^{p-1}).$$

The idea

In other words, we simply need to count the dimensions of the kernel and image of each matrix, δ.

Computing $H^{\bullet}(S^1 \wedge S^1, \mathbb{R})$: The Cech cohomology groups

$$0 \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} 0 \qquad \delta(\overline{f})_{ij} := f_j - f_i$$

- $H^{-1}(S^1 \wedge S^1, \mathbb{R}) = 0$
- $H^0(S^1 \wedge S^1, \mathbb{R}) = ker(\delta^0)/im(\delta^{-1}) \cong \mathbb{R}^{1-0} = \mathbb{R}^1$
- $H^1(S^1 \wedge S^1, \mathbb{R}) = ker(\delta^1)/im(\delta^0) \cong \mathbb{R}^{6-4} = \mathbb{R}^2$
- $H^2(S^1 \wedge S^1, \mathbb{R}) = 0$

Presentation Outline

Basic summary of Cech cohomology computations

1. Fix a topological space, X.
2. Choose a (good) open cover.
3. Write down (and draw) the Cech nerve of this cover.
4. Write out your Cech complex.
5. Write out the boundary maps (matrices).
5. Compute the kernel and image subspaces of these matrices by the Gauss-Jordan algorithm and the Rank-Nullity theorem.
6. Compute the cohomology groups by subtracting dimension numbers.
7. State your computed cohomology groups.

Linear Algebra Prerequisites and Applications

- Computing the matrix for a given linear map.
- The kernel and image subspaces for a linear map.
- Rank/Nullity Theorem
- Gauss-Jordan Reduction
- Dimension of the quotient of a finite-dimensional vector space by a subspace.

Presentation Outline

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CJM

Some simple candidates for your space, X.

- The circle, S^1.
- The wedge of two circles, $S^1 \wedge S^1$.
- The sphere, S^2.
- The torus, $S^1 \times S^1$. (a nice independent project)

Presentation Outline

An algorithm for computing cohomology theories for a topological space, X

Cech Cohomology computes . . . under the right conditions . . .

- singular cohomology (Topology)
- de Rham cohomology (Differential Geometry)
- sheaf cohomology (Cech-DeRham Cohomology; Algebraic Geometry)
- ... perhaps more things the speaker doesn't fully comprehend.

Interesting Applications

Cech Cohomology is ideally used in . . .

- Topological Data Analysis . . . in particular, in Persistent Homology
- ??

Thank you!

Questions?