# A thematic linear algebra course focused on four problems of the form $T(x) = b$

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# Linear algebra at Ferris State

Our linear algebra course is mostly comprised of three distinct groups of students:

1. (Applied) math majors (modeling and ODEs)
2. Actuarial science majors (Markov chains)
3. Surveying engineering majors (least-squares methods)

When teaching this course in 2014, my students...

- were very competent at performing standard linear algebra computations, but
- did not demonstrate understanding of course concepts, and
- didn't engage well with segments of the course not tailored to their interests.

# My proposed solution

When I next taught linear algebra (2016), I decided to rearrange its content.

I built a course centered around four motivating problems, tailored to the various groups taking linear algebra at Ferris.

# The four problems

## 1. A surveying problem

A surveyor marks ten points on a plot of land, and uses instruments to record the elevation difference between several pairs of these ten points. This gives a system of linear equations, each of the form

$$x_i - x_j = const$$

where $x_i$ is the elevation at the $i^{th}$ point.

Goal is to find (i.e. estimate with minimal error) the elevation at each point, assuming the elevation at the first point is known for certain.

(The problem is rigged so that it clearly has no solution.)

# The four problems

## 2. A traditional system of linear equations

I use a system which would be solved to find the stationary distribution of a Bonus-Malus system.

Bonus-Malus systems are Markov models used in actuarial science to determine how much to discount premiums for customers who have no claims in a given time period.

# The four problems

## 3. A linear differential equation (or equations)

$$y'(t) + 5y(t) = 14$$

$$y''(t) - 3y'(t) + 10y(t) = \cos t$$

I advertise these as being related to things like population dynamics, heating/cooling problems, mass-spring systems or electrical circuits, without giving much exposition.

# The four problems

## 4. An indefinite integral

$$\int \cos x \, dx$$

# Course outline

**1. Introduce the four problems, and announce a goal of understanding why these four problems are essentially the same.**

**Early student observations:**

1. all four problems can be formulated as one or more equations;
2. the equations typically contain addition and/or (scalar) multiplication.

**2. Develop machinery needed to understand the similarities between these problems.**

This means vector spaces and subspaces (not just $\mathbb{R}^n$, but spaces of matrices, spaces of functions, sequence spaces, etc.), with lots of geometric intuition: equations of lines and planes, etc.

Once these concepts were introduced, students were able to observe that all four problems could be viewed as equations of the form $T(x) = b$, where $b$ is a vector and $T$ is a function from one vector space to another.

**2. Develop machinery needed to understand the similarities between these problems.**

This means vector spaces and subspaces (not just $\mathbb{R}^n$, but spaces of matrices, spaces of functions, sequence spaces, etc.), with lots of geometric intuition: equations of lines and planes, etc.

This leads naturally to a discussion of what is going on with the $T$: linear transformations (not just matrix multiplication, but differentiation of functions, geometric transformations, etc.)

**3. Solve the equation $T(x) = b$**

and apply the generic solution to the four key problems.

The generic solution of $T(x) = b$, is of course, $x = x_p + ker(T)$ (provided $b \in im(T)$).

In this setting, Gaussian elimination is presented as a tool to reveal whether or not $b \in im(T) = C(A)$, give a particular solution $x_p$, and a basis of $ker(T) = N(A)$.

Projections and least-squares methods are then developed as a tool to handle the situation where $b \notin im(T)$.

# Pedagogy

- Traditional lectures
- Group work (often inquiry-based)
- Frequent review, with an emphasis on how recent material relates to the big picture.
- Homework
- Exams

# Impacts

## Improvement in student performance in linear algebra

Some data from my linear algebra courses at Ferris State:

| | traditional approach ($n = 28$) | four problems approach ($n = 18$) |
|---|---|---|
| Median course average | 79.3 | 86.1 |
| Percentage of students finishing course with 90% average or better: | 14% | 44% |
| Percentage of students finishing course with 60% average or better: | 82% | 100% |

# Impacts

## Improved student performance in differential equations

| | traditional approach ($n = 51$) | four problems approach ($n = 9$) |
|---|---|---|
| Median course average in differential equations | 79.3 | 86.1 |
| Percentage of students finishing differential equations course with 90% average or better: | 37% | 66% |

# Impacts

## Students are happier

Average student course ratings (5-point scale):

| | |
|---|---|
| traditional approach: | 4.1 |
| four problems approach: | 4.7 |

## I'm happier

The course material flows nicely; abstract concepts have a natural reason for their introduction.

Students are more engaged in class.

Students learn concepts without having a computational crutch to rely on (example: linear independence)