Solving a System of Linear Equations Using Ancient Chinese Methods

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Outline

Question

History Question

Who invented Gaussian Elimination?

When was Gaussian Elimination developed?

- Carl Friedrick Gauss lived from 1777-1855.
- Arthur Cayley was one of the first to create matrix algebra in 1858.

Ancient Chinese Origins

Juizhang Suanshu

The Nine Chapters on the Mathematical Art is an anonymous text compiled during the Qin and Han dynasties 221 BCE - 220 AD. It consists of 246 problems and their solutions arranged in 9 chapters by topic.

Fangcheng

Chapter 8 of the *Nine Chapters* is translated as Rectangular Arrays. It concerns the solution of systems of linear equations.

Liu Hui

The Chinese mathematician Liu Hui published an annotated version of *The Nine Chapters* in 263 AD. His comments contain a detailed explanation of the *Fangcheng* Rule

Counting Rods

The ancient Chinese used counting rods to represent numbers and perform arithmetic. The digits 1-9 were represented as shown.

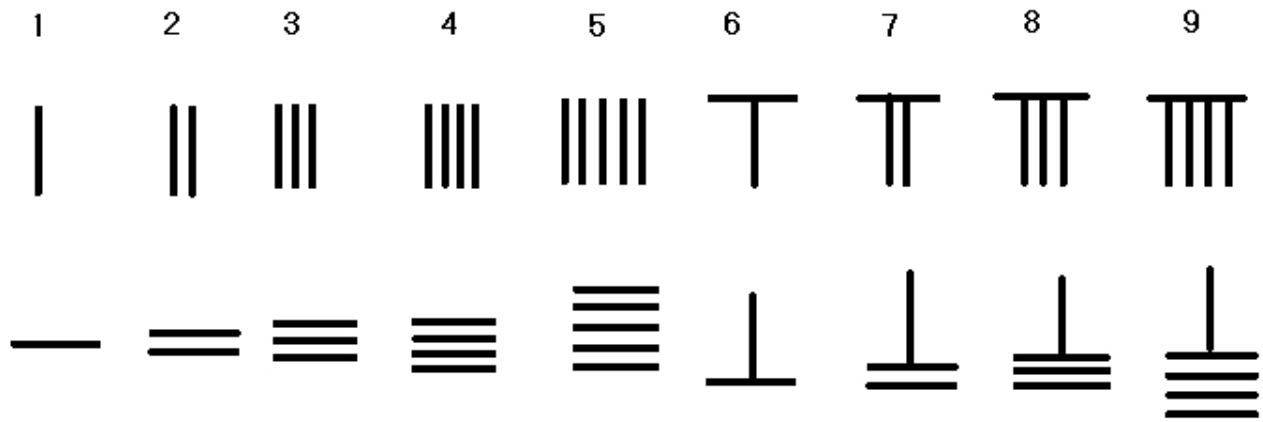

Figure: Vertical and Horizontal Counting Rod Numerals

Numbers

Counting Rod Numerals

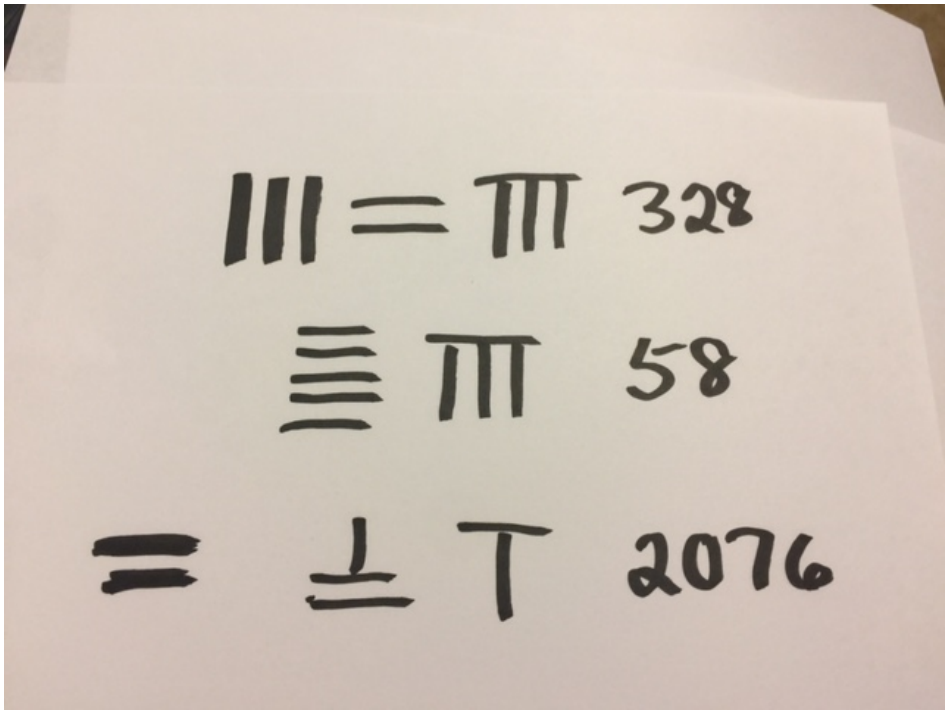

Figure: Examples of Rod Numbers

Arithmetic on a Counting Board

- The Chinese algorithms involved physically manipulating the rods on a flat surface (counting board)
- Numbers were arranged with place values in columns.
- Systems of linear equations were viewed more as division problems than equations.
- The array notation seems like a natural extension of the techniques for arithmetic.
- The power of finding an efficient notation is evident in this technique.

Chapter 8 Problem 1

Problem 1

Now given 3 bundles of top grade paddy, 2 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 39 *dou* of grain. 2 bundles of top grade paddy, 3 bundles of medium grade paddy, [and] 1 bundle of low grade paddy, yield 34 *dou*. 1 bundle of top grade paddy, 2 bundles of medium grade paddy, [and] 3 bundles of low grade paddy, yield 26 *dou*. Tell: how much paddy does one bundle of each grain yield? Answer: Top grade paddy yields $9\frac{1}{4}$ *dou* [per bundle]; medium grade paddy $4\frac{1}{4}$ *dou*; [and] low grade paddy $2\frac{3}{4}$ *dou*.

The Rule

The *Fangcheng* Rule

[Let Problem 1 serve as example.] Lay down in the right column 3 bundles of top grade paddy, 2 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 39 *dou* of grain. Similarly for the middle and left column. Use [the number of bundles of] top grade paddy in the right column to multiply the middle column then merge. Again multiply the next [and] follow the pivoting. Then use the remainder of the medium grade paddy in the middle column to multiply the left column and pivot. The remainder of the low grade paddy in the left column is the divisor, the entry below is the dividend. The quotient is the yield of low grade paddy. . . .

The Array

Chinese was read from top to bottom and right to left. The Chinese array for Problem 1 is as follows:

1	2	3
2	3	2
3	1	1
26	34	39

The modern augmented matrix

$$\begin{pmatrix} 3 & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{pmatrix}$$

Transform the Array

Following the part of the *Fangcheng* Rule printed on the last slide transforms the array into the following triangular form.

		3
	5	2
36	1	1
99	24	39

Contrast this with the augmented matrix in upper triangular form:

$$\begin{pmatrix} 3 & 2 & 1 & 39 \\ 0 & 5 & 1 & 24 \\ 0 & 0 & 36 & 99 \end{pmatrix}$$

My PSP

Goals

- I wrote a Primary Source Project introducing Gaussian elimination from *The Nine Chapters* followed by a treatment using modern notation.
- The project replaces the first section or two of a standard linear algebra text on elimination and elementary row operations.
- My main goal is to introduce the material in an interactive and novel way that grabs students' attention.
- The historical context levels the playing field as the material is unfamiliar to all.
- My secondary goal is to introduce the challenge of efficient computation and the power of efficient notation.

Pedagogy

- Students read excerpts from a translation of the original text and the commentary by Liu Hui.
- The PSP text guides the students through the reading with extra exposition and questions.
- The PSP is designed for individual reading assignments and in-class group work on the exercises.
- Notes to Instructor contain suggested timelines for implementation and suggestions for modifying the material to fit the goals of your course.

Outline of the PSP

- Historical Background - short
- Chinese Counting Rod Arithmetic and the Sign Rule
- Setting Up the Array using the Fangcheng Rule instructions
- Elimination - reducing the array to triangular and modern elimination notation
- Substitution - Two versions of the Chinese substitution steps along with modern back substitution
- A Comparison of the Arithmetic

Comparing Arithmetic

I have the students work Problems 1, 3, 7 and 8 from the *Nine Chapters* and a problem posed in modern language. They are asked to do the arithemtic for each problem in multiple different ways, and then asked which technique was easier. Since the Chinese problems have answers involving fractions, this is a nontrivial exercise. The substitution method laid out in the Fangcheng Rule is not intuitive, but has the advantage of delaying the need for fractions until the last step in most cases, a very useful algorithm for hand calculations.

My Experience

The Interview Talk

The Fangcheng Rule was my topic of conversation in the linear algebra class I was asked to teach as a part of my interview for my current position. (Tenure pending NOW).

History of Math

I used a preliminary draft of this PSP as a 1-2 day exercise in my history of math course in 2016 and the students really liked discovering the similarities between modern notation and the ancient Chinese text.

Linear Algebra

I implemented this PSP in my Fall 2017 Linear Algebra Course in a 2.5 week format. I will be using this material in my Spring 2018 Linear Algebra course.

THe TRIUMPHS Collaborative Project

- Transformating Instruction in Undergraduate Mathematics via Primary Historical Sources
- The goal is to write, test and disseminate high-quality curriculum for a regular math class that teaches topics through guided reading of original sources.
- Primary Source Projects (PSPs) are reviewed by two readers and revised before being available to the community.
- PSPs are tested in classrooms by instructors like yourselves.
- Mine was tested by 2 other instructors in the fall of 2017, and will be used by myself and one additional tester this spring.

The TRIUMPHS PIs

TRIUMPHS Team

- Dominic Klyve - Central Waskington University
- Janet Heine-Barnet - University of Colorado at Pueblo
- Diana White - University of Colorado Denver
- Jerry Lodder - New Mexico State University
- Kathleen Clark - Florida State University
- Danny Otero - Xavier University
- Nicholas Scoville - Ursinus College

Student Response

Comments from other instructors

- The students felt satisfied that they were able to solve the ancient problems and relate them to math they were familiar with.
- The PSP eliminated the inoculation effect where some students think they know the material already.

My Students

- Students found it fascinating that the ancient Chinese discovered this method.
- Students were engaged in the group work, one commented on the evaluation form that this was her favorite part of the course for that reason.
- Students grasped the technique more thoroughly than normal for my student population and performed well on the test.

Instructor Comments

- I enjoyed knowing some of the historical background.
- Breaking students into groups is not a normal part of my teaching, but I tried it here.
- I appreciate the detail of the PSP, starting with counting rods, interpreting the original problem and then relating it to math they already know.
- My goal is to engage my class in active inquiry-based learning, and this PSP was in line with how I am teaching my class.

Acknowledgements

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Opportunity for You to Try a PSP

TRIUMPHS Website

`http://webpages.ursinus.edu/nscoville/TRIUMPHS.html`

Site Tester

The TRIUMPHS team is offering opportunities for instructors to participate in the research by officially testing a PSP in their classroom and participating in the research on their effectiveness.

TRIUMPHS PIs Available in San Diego

The TRIUMPHS Project Poster in the MAA Poster Session Projects Supported by the NSF is from 2-4 this afternoon in Exhibi Hall B2, Ground Level.