Linear Algebra in Digital World

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Mathematics and Animation

- Dinosaurs of the Jurassic Park
- Wonders of the Lord of the Rings
- Star turn of Gollum
- 3D computer games

Bézier Curves

- A polynomial is said to interpolate a set of data points if it passes through those points
- Bézier curve is an example which interpolates only first and last points
- Smooth and specified in terms of few points
- Bézier curves are independent of its dimension

Formation of Bézier Curves

P. Bézier got the idea during his work for Renault automobile company.

Any point on a curve must be given by a parametric function of the following form

$$p(u) = \sum_{i=0}^{n} p_i \, C_i(u) \text{ for } u \in [0,1]$$

p_i → n+1 vertices

$C_i(u)$ → basis functions

Matrix form of a Bézier Curve

If C_i are Bernstein polynomials,

$$\boldsymbol{p}(u) = \begin{bmatrix} p_0 & p_1 & p_2 & p_3 \end{bmatrix} \begin{bmatrix} B_0^3(u) \\ B_1^3(u) \\ B_2^3(u) \\ B_3^3(u) \end{bmatrix}$$

$$\boldsymbol{p}(u) = \begin{bmatrix} p_0 & p_1 & p_2 & p_3 \end{bmatrix} \begin{bmatrix} (1-u)^3 \\ 3(1-u)^2 u \\ 3(1-u)u^2 \\ u^3 \end{bmatrix}$$

Matrix form and Monomials

A cubic Bézier curve

$$p(u) = (1-u)^3 p_0 + 3(1-u)^2 u p_1 + 3(1-u) u^2 p_2 + u^3 p_3$$

Monomials $1, u, u^2, u^3$

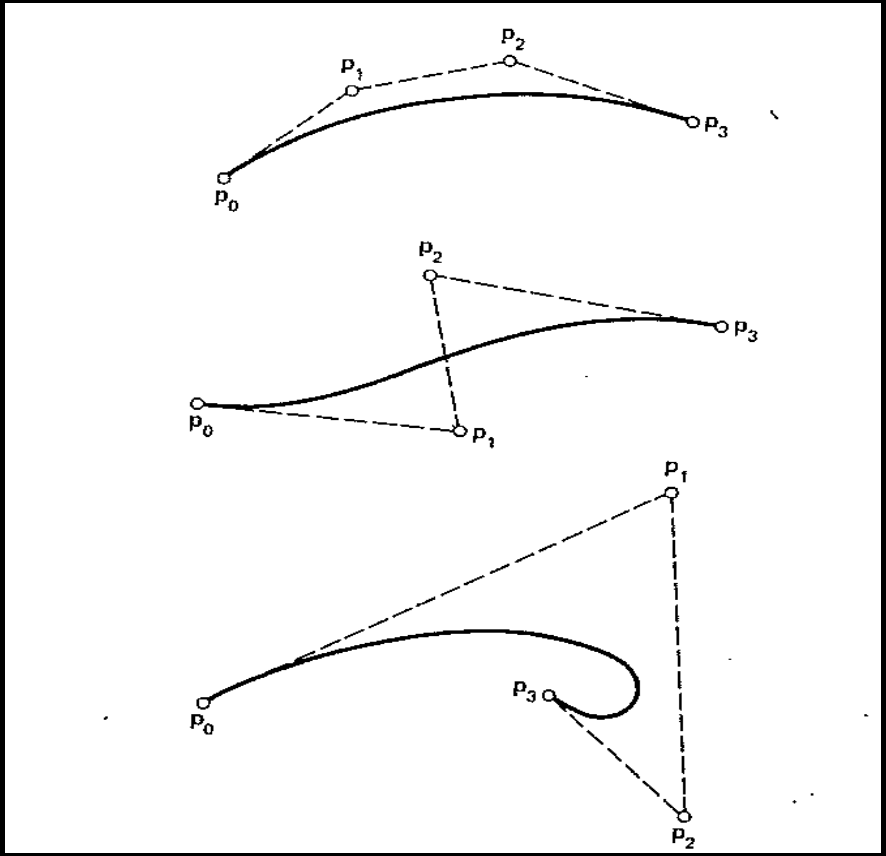
$$p(u) = \begin{bmatrix} p_0 & p_1 & p_2 & p_3 \end{bmatrix} \begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ u \\ u^2 \\ u^3 \end{bmatrix}$$

Properties of Bézier Curves

Properties

- $\boldsymbol{p}(u)$ → four control points
- Starts from p_0 in the direction of p_1
- Ends at p_3 coming from the direction of p_2
- Only two control points lie on $\boldsymbol{p}(u)$
- Other 2 control points pull on the curve
- $\boldsymbol{p}(u)$ acts as a flexible and stretchable curve

Cubic Bézier Curves



Properties

Degree 3 Bézier curve can be expressed as

$$p(u) = B_0(u)p_0 + B_1(u)p_1 + B_2(u)p_2 + B_3(u)p_3$$

The derivative

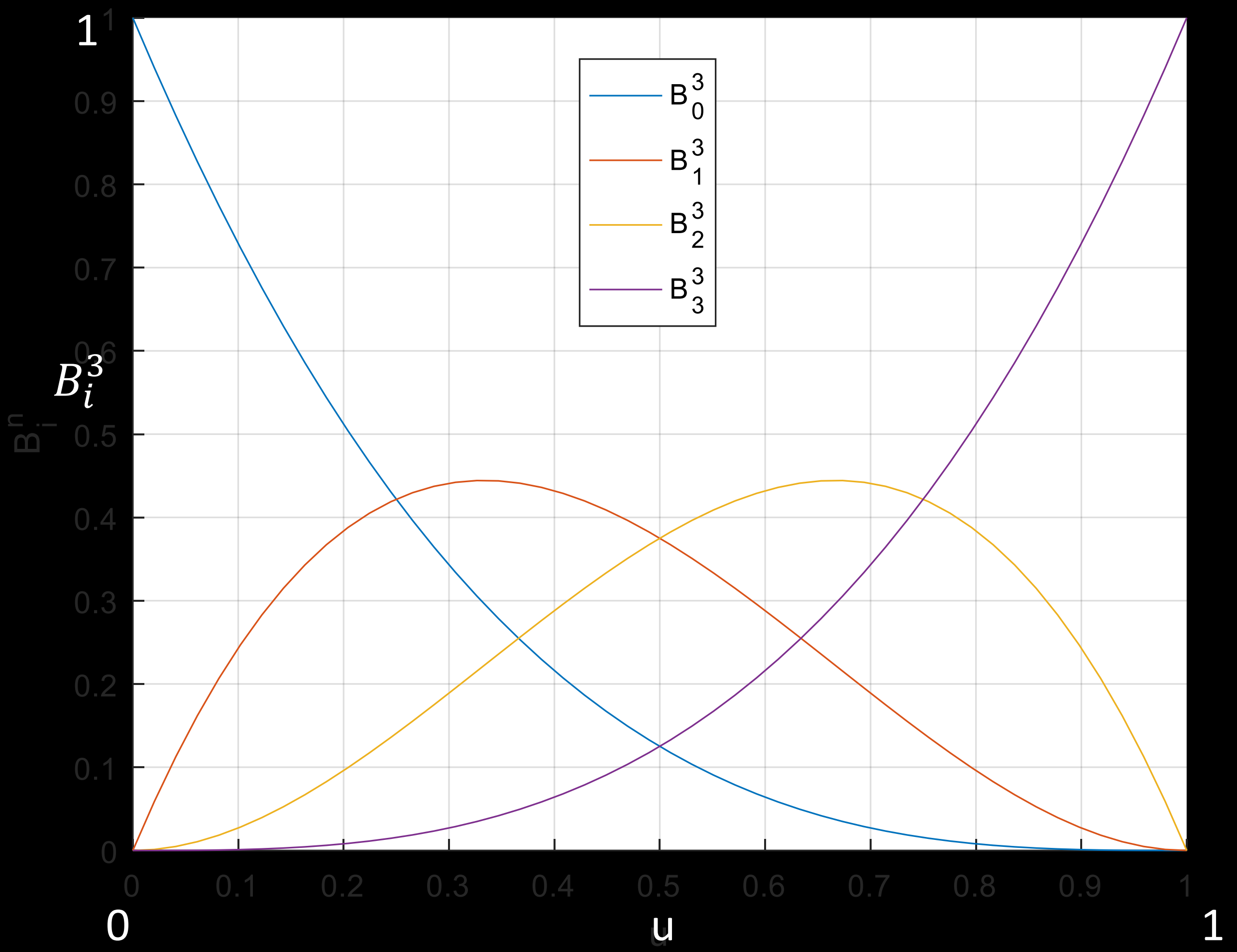
$$p'(u) = B'_0(u)p_0 + B'_1(u)p_1 + B'_2(u)p_2 + B'_3(u)p_3$$

At the beginning: $p'(0) = 3(p_1 - p_0)$

At the end $p'(1) = 3(p_3 - p_2)$

Bernstein polynomial affects the control point

- At u=0, weight of p_0=1
- At u=1/3, weight of p_1= max
- At u=2/3, weight of p_2 = max
- At u=1, weight of p_3 =1
- Sum of four functions are always one



Degree Elevation

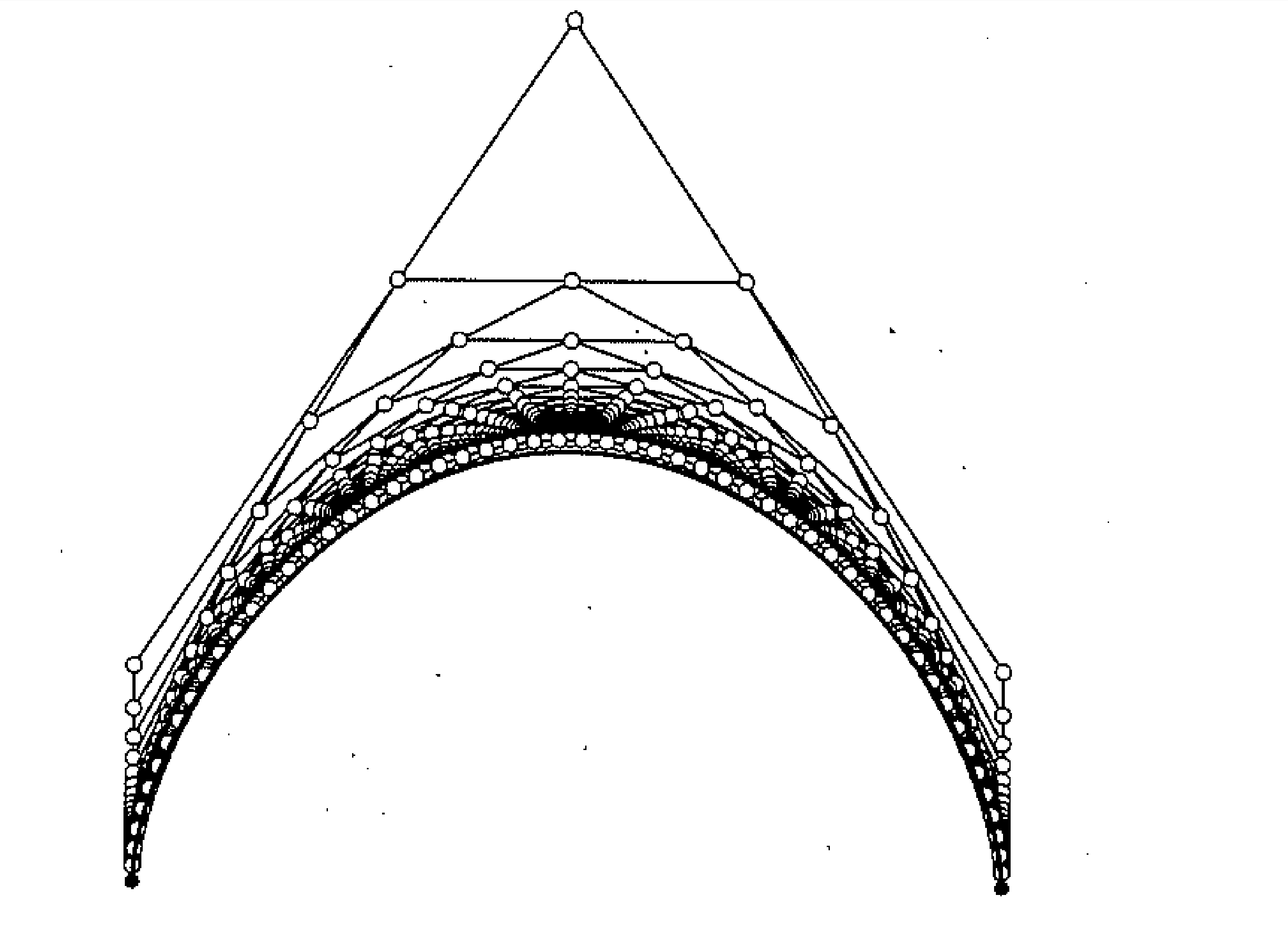
$$p(u) = (1-u)^2 p_0 + 2(1-u)u p_1 + u^2 p_2$$

Trick is to multiply by $[u + (1-u)]$

$$p(u) = (1-u)^3 p_0 + 3(1-u)^2 u \left[\frac{1}{3} p_0 + \frac{2}{3} p_1\right] + 3(1-u)u^2 \left[\frac{2}{3} p_1 + \frac{1}{3} p_2\right] + u^3 p_2$$

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 2/3 & 0 \\ 0 & 2/3 & 1/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix}$$

$$C = DP$$



A degree four Bézier curve, degree elevated up to degree 50

Degree Reduction

In degree elevation, we were given P, we obtained C.

$C=DP$

In Degree reduction, C is given, Find P

$DP=C$

$$D^T DP = D^T C$$

We have a linear system for the unknown P, with a square coefficient matrix $D^T D$.

Solution is straightforward $D^T D$ is invertible

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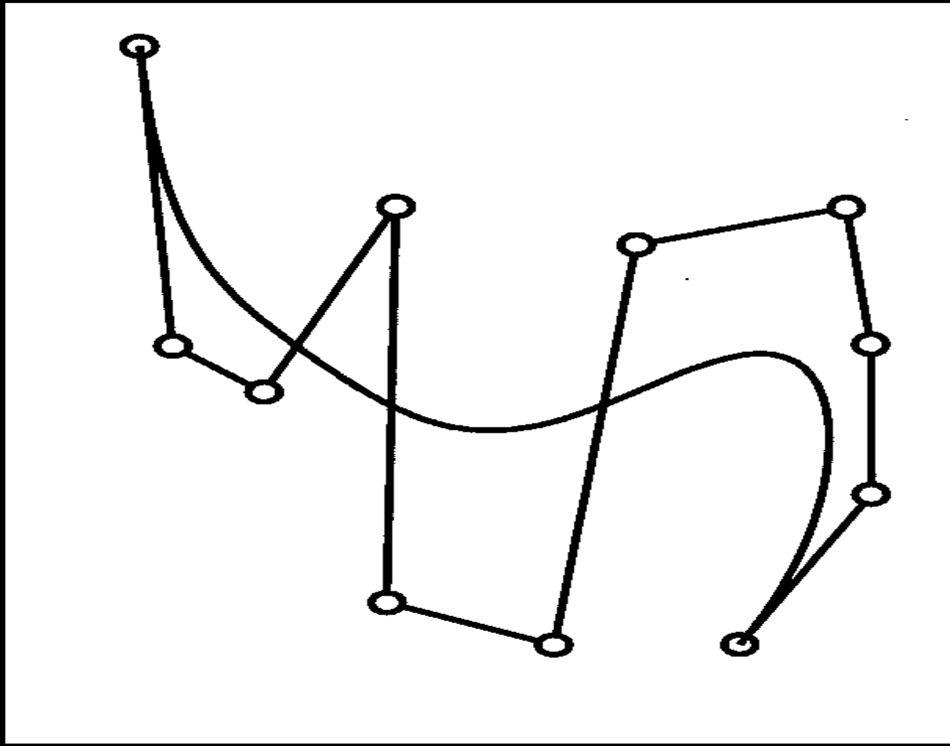
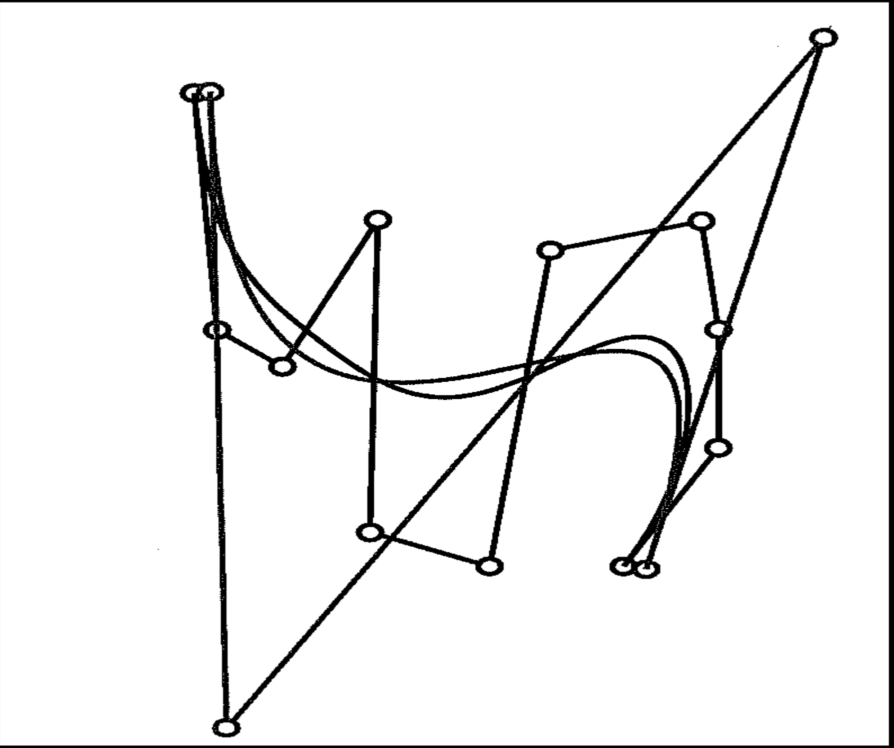
An example

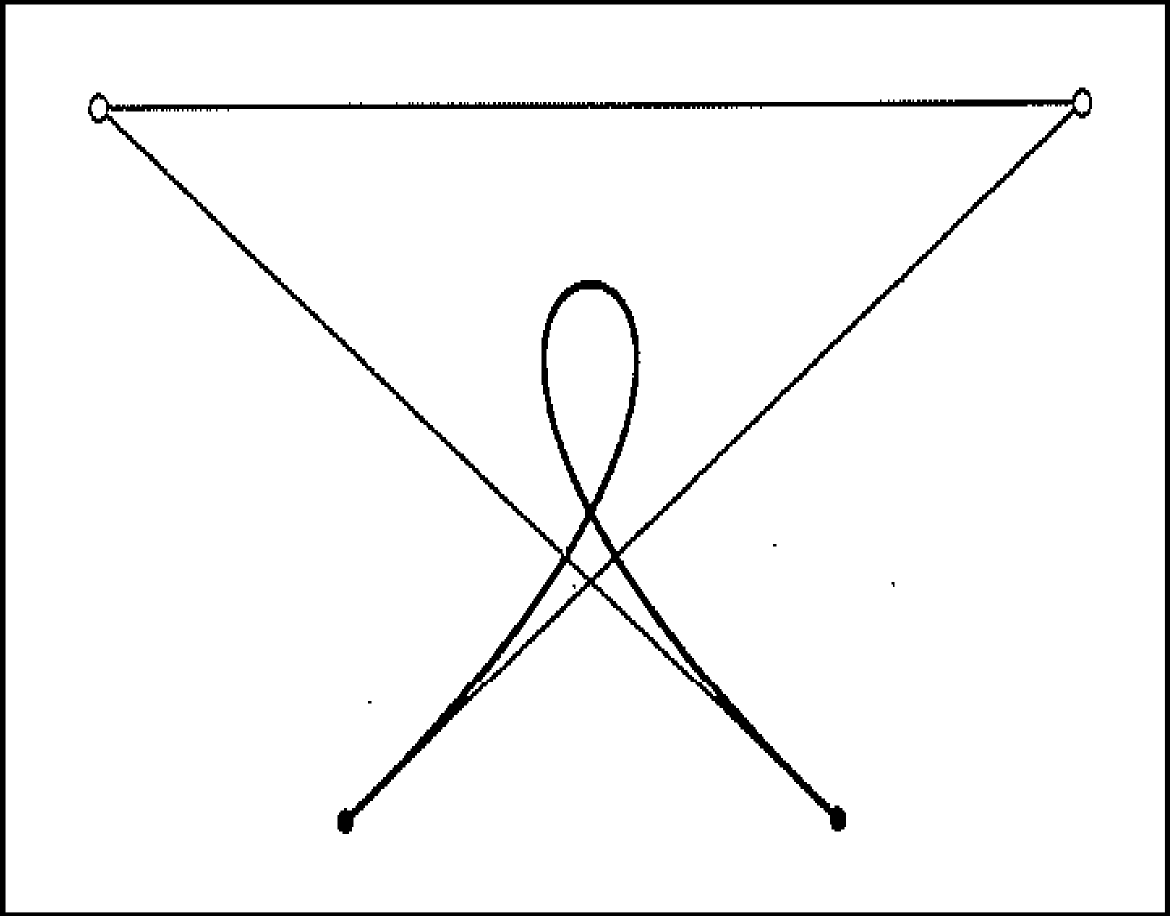
Degree 10



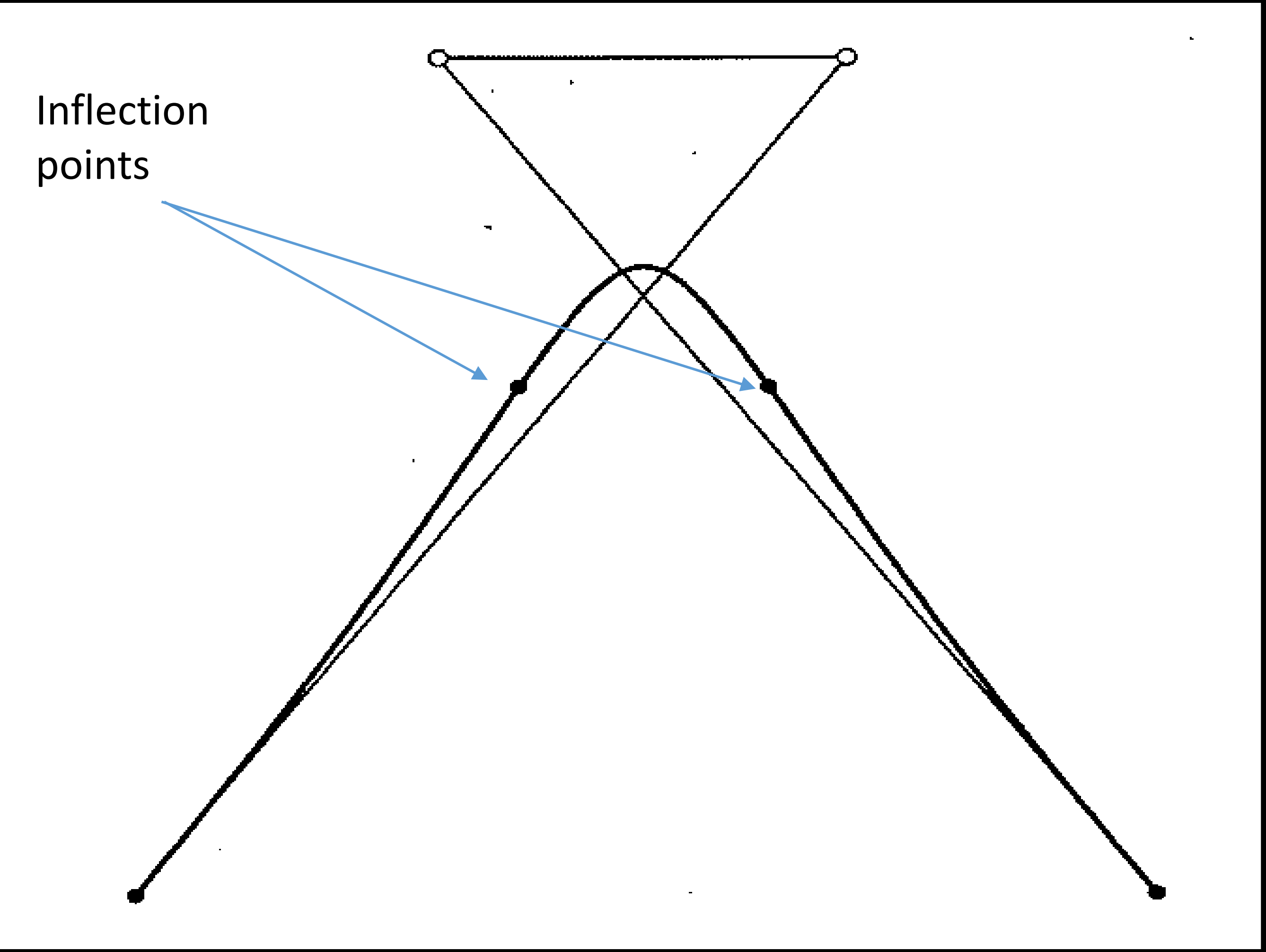
Degree 3



Self –intersecting Bézier curve



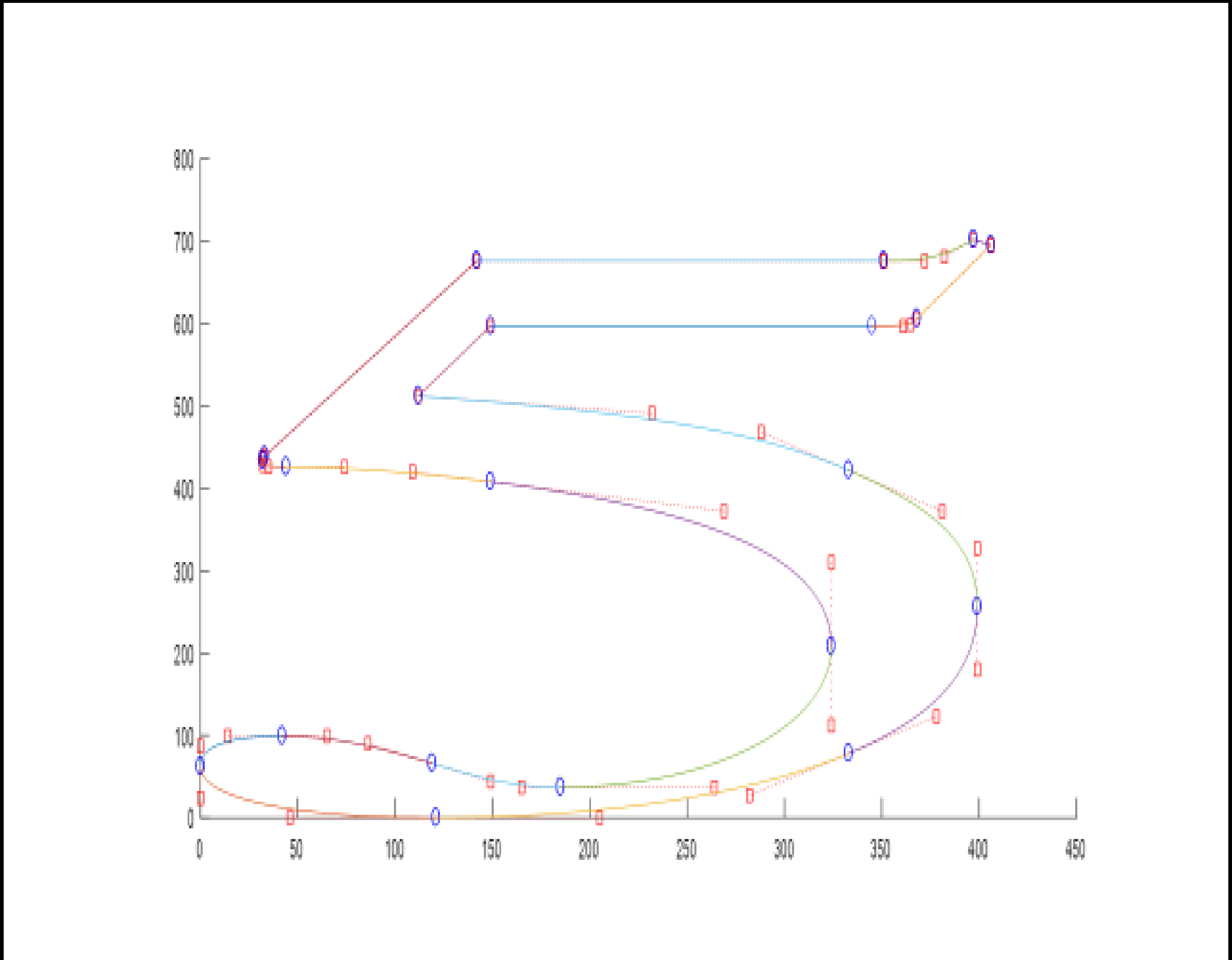
A Bézier curve with 2 inflection points



Piecewise Bézier Curves

- A single degree three Bézier curve has only a limited range of shapes
- Our letters and numbers are more complicated than a single degree 3 Bézier curve
- Higher degree Bézier curve can be used to draw more complicated curves, but it is not easy to work with
- It is better to combine multiple Bézier curves to form a complicated curve

Example



800
700
600
500
400
300
200
100
0
0
50
100
150
200
250
300
350
400
450

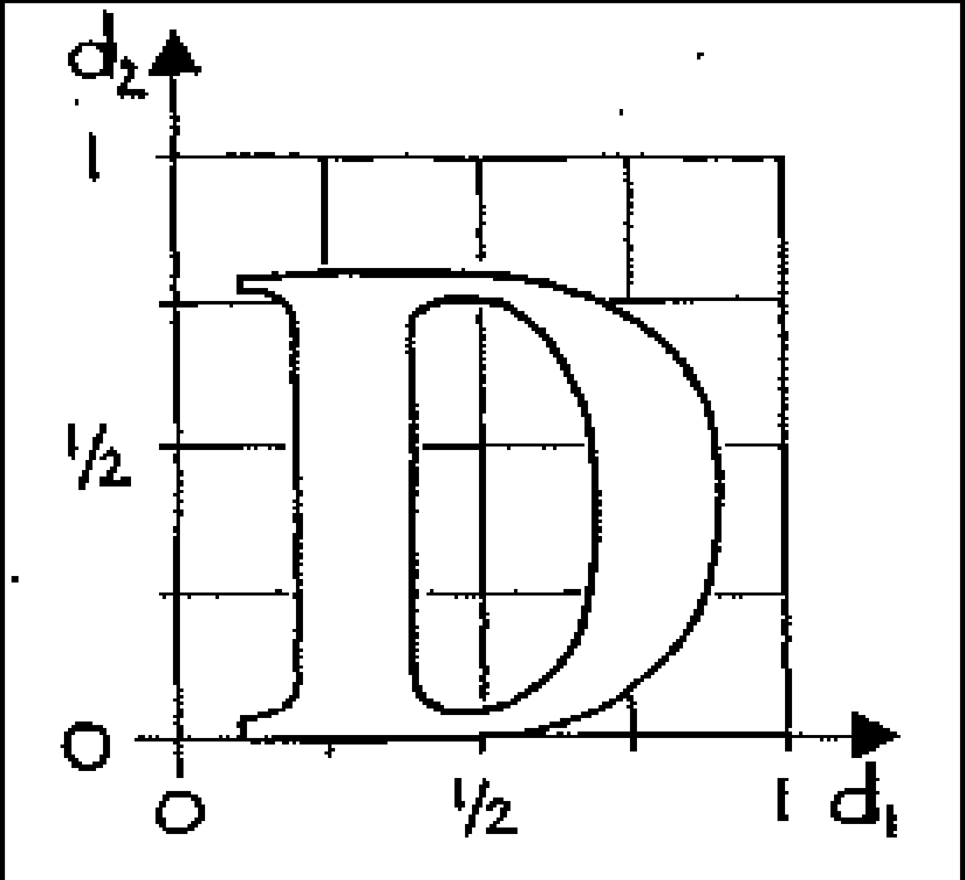
Benefits of using Bézier curves

- Letters and numbers → piecewise Bézier curves

- Easy to rescale (Application of Linear Algebra)

- If pixel maps of the fonts are used,
resizing → aliasing effects.

Local Coordinates

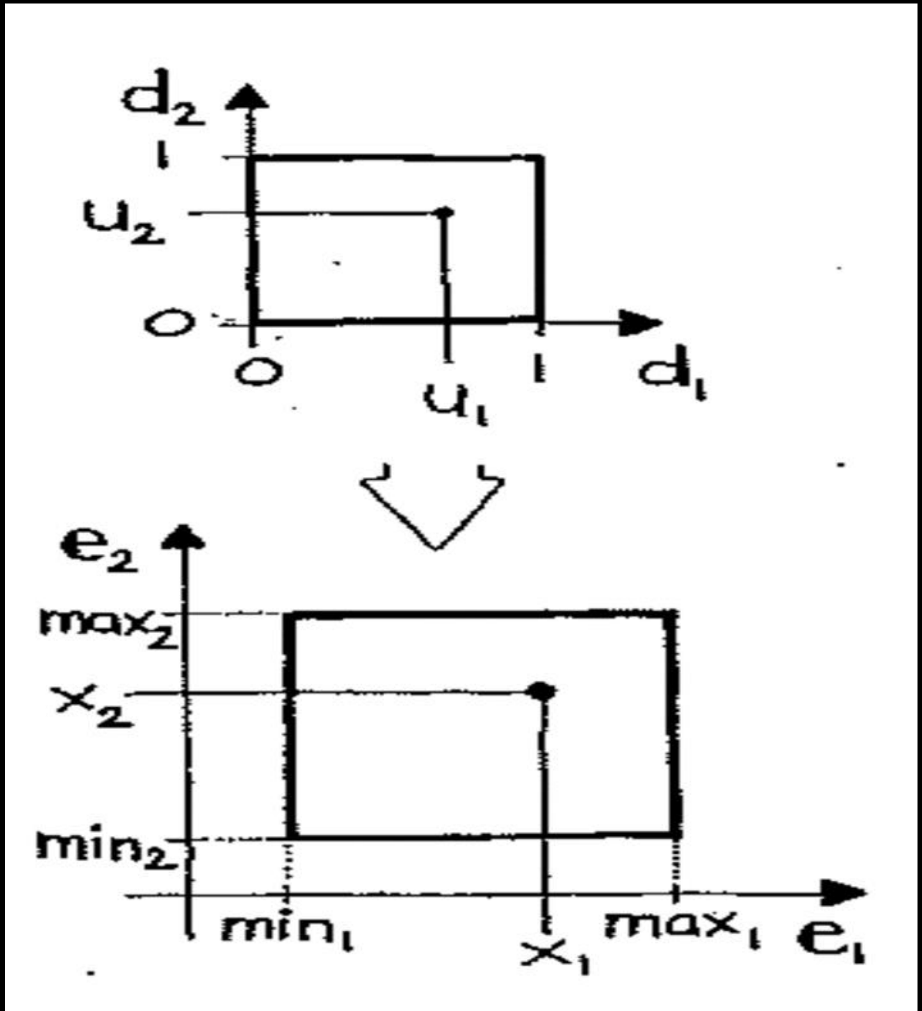


Local to Global

- From local coordinate system (d) to global coordinate system (e)

- Change of basis

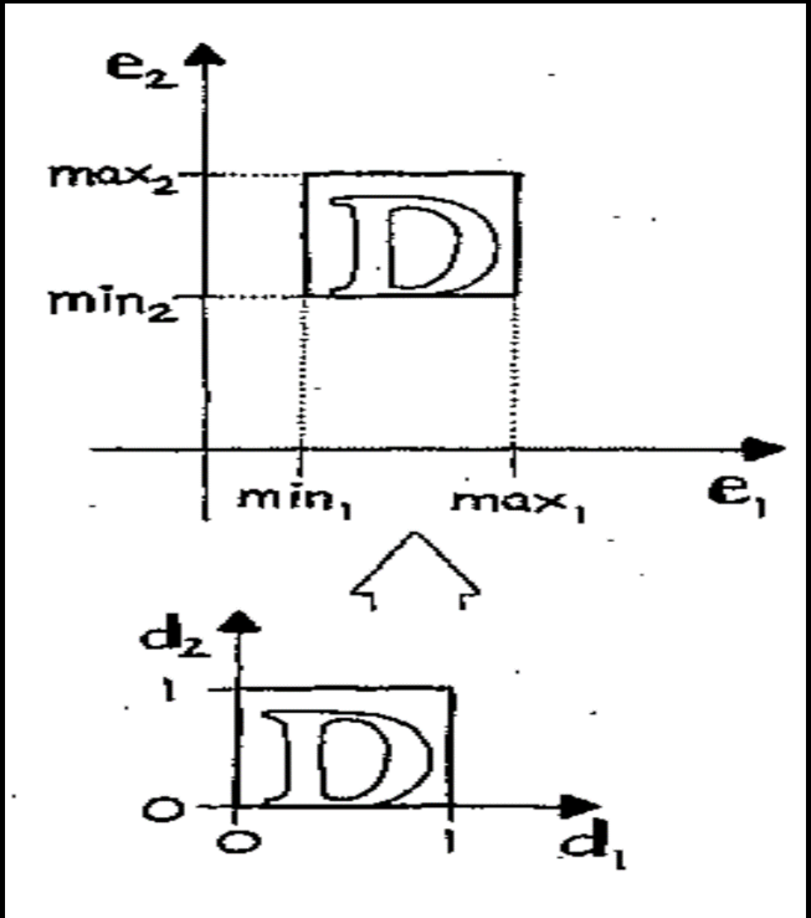
- Local interval [0, 1] to global interval $[min_1, max_1]$



Matrix form

- $$\frac{d_1-0}{1-0}=\frac{e_1-min_1}{max_1-min_1}$$
- $e_1 = min_1 + (max_1 - min_1)d_1 \quad ; \qquad d_2 \rightarrow e_2$
- $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} min_1 \\ min_2 \end{bmatrix} + \begin{bmatrix} max_1 - min_1 & 0 \\ 0 & max_2 - min_2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$

D is mapped from local to global coordinates



Initial stage of an animated character

- Model of a dog
- Hi tech Animated movie
- The dog → coordinates
- CMM (coordinate measuring machine)
- CMM's arm touch → *x, y, z* coordinates
- Repeated hundred times
- This process is called digitizing

References

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Thank You