# Exploring Subspaces and Bases Through Magic Squares

Michelle Ghrist

Department of Mathematics
Gonzaga University

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# Outline

# Magic Squares

Magic squares have been studied for over 2600 years.

## Connections to many fields, e.g.:

- Abstract algebra
- Combinatorics and graph theory
- Mathematical art
- Generating functions
- Splines
- Ciphers
- Elliptic curves
- Number theory

# Context

## The Course (U. S. Air Force Academy)

- Proof-based linear algebra course
- Most students are math majors
- Textbook: Stephen Lay's Linear Algebra and Its Applications
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Original project was given as an extra credit assignment but can easily be extended.

# The Inspiration

**From Gilbert Strang's Linear Algebra and Its Applications**

"In the space of all 2 by 2 matrices, find a basis for the subspace of matrices whose row sums and column sums are all equal.

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(Extra credit: Find five linearly independent 3 by 3 matrices with this property.)"

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Elements of this subspace are often called *(non-normal) semimagic squares*.

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Goal: Have students work with definitions of vector space, subspace, dimension, and basis.

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## What I should have assigned

- Show that the set of all 3x3 (non-normal) magic squares is a subspace of $M_{3x3}$.
- Find the dimension of this subspace.
- Find a basis for the set of all 3x3 (non-normal) magic squares.

# Student solution, part 1

Solve the linear system:

$$a_{k1}+a_{k2}+a_{k3} = a_{1k}+a_{2k}+a_{3k} = a_{11}+a_{22}+a_{33} = a_{31}+a_{22}+a_{13} = T$$

for all $k = 1, 2, 3$ and some $T$ (8 equations in 10 unknowns).

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In matrix form:

$$\left[\begin{array}{cccccccccc|c}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0
\end{array}\right]$$

# Student solution, part 2

This row reduces to

$$\left[\begin{array}{cccccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2/3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 1/3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -2 & 2/3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & -4/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

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0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1/3 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & -4/3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

so there are 3 free variables. The dimension of this subspace is 3 and a basis is

$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

# Student solution, part 3

Using our basis, we can confirm Édouard Lucas's $19^{th}$-century result that all 3x3 magic squares are of the form

$$\begin{bmatrix} c-b & c+(a+b) & c-a \\ c-(a-b) & c & c+(a-b) \\ c+a & c-(a+b) & c+b \end{bmatrix}$$

# The 4x4 problem

Solve the linear system

$$\sum_{i=1}^{4} a_{k,i} = \sum_{i=1}^{4} a_{i,k} = \sum_{i=1}^{4} a_{i,i} = \sum_{i=1}^{4} a_{i,5-i} = T$$

for all $k = 1, 2, 3, 4$ and some $T$ (10 equations in 17 unknowns).

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for all $k = 1, 2, 3, 4$ and some $T$ (10 equations in 17 unknowns). The dimension of this subspace is 8 and a basis is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

# General problem, part 1

Solve the linear system:

$$\sum_{i=1}^{n} a_{k,i} = \sum_{i=1}^{n} a_{i,k} = \sum_{i=1}^{n} a_{i,i} = \sum_{i=1}^{n} a_{i,n+1-i} = T$$

for all $k = 1, 2, \ldots n$ and some $T$.

There are $2n + 2$ equations in $n^2 + 1$ unknowns.

# General problem, part 2

| $n$ | dim(magic squares) | dim(semimagic squares) |
|---|---|---|
| 2 | 1 | 2 |
| 3 | 3 | 5 |
| 4 | 8 | 10 |
| 5 | 15 | 17 |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $n$ | | |

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$$\sum_{k=1}^{n} \left( \sum_{i=1}^{n} a_{k,i} \right) = \sum_{k=1}^{n} \left( \sum_{i=1}^{n} a_{i,k} \right) = nT$$

See Ward (1980) for proof that above formulas are correct.

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## Some questions

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- What if we restrict to circulant or symmetric matrices?

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- If $M$ is magic, is its adjoint also magic?
- If $M$ is magic and nonsingular, is its inverse also magic?
- Is the product of two magic squares also magic?
- What if we restrict to circulant or symmetric matrices?
- What if we restrict entries to squares or primes?
- How many *normal magic squares* are there of order $n$? (entries unique: $1, 2, \ldots, n^2$)

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- More than two dimensions
- Magic circles, spheres, (tori?)
- Latin squares; Euler squares

# Conclusions

- Magic squares and their many variations provide rich opportunities to learn and apply mathematical ideas.
- Homework problems and projects involving magic squares can be created with many various lengths, depths, and difficulties.
- There are still many open questions about magic squares.

Thank you for coming!

Michelle Ghrist, ghrist@gonzaga.edu

# Some references


- **Vector Spaces of Magic Squares**, James E. Ward III. *Mathematics Magazine* (1980), 108-111.
- **The Lost Squares of Dr. Franklin: Ben Franklin's Missing Squares and the Secret of the Magic Circle**, Matthias Beck, Moshe Cohen, Jessica Cuomo, and Paul Gribelyuk. *The American Mathematical Monthly*, 108(6) (2001), 489-511.
- **The Number of "Magic" Squares, Cubes, and Hypercubes**, Paul C. Pasles. *The American Mathematical Monthly*, 1110(8) (2003), 707-717.


Michelle Ghrist, ghrist@gonzaga.edu