Di-eigenals

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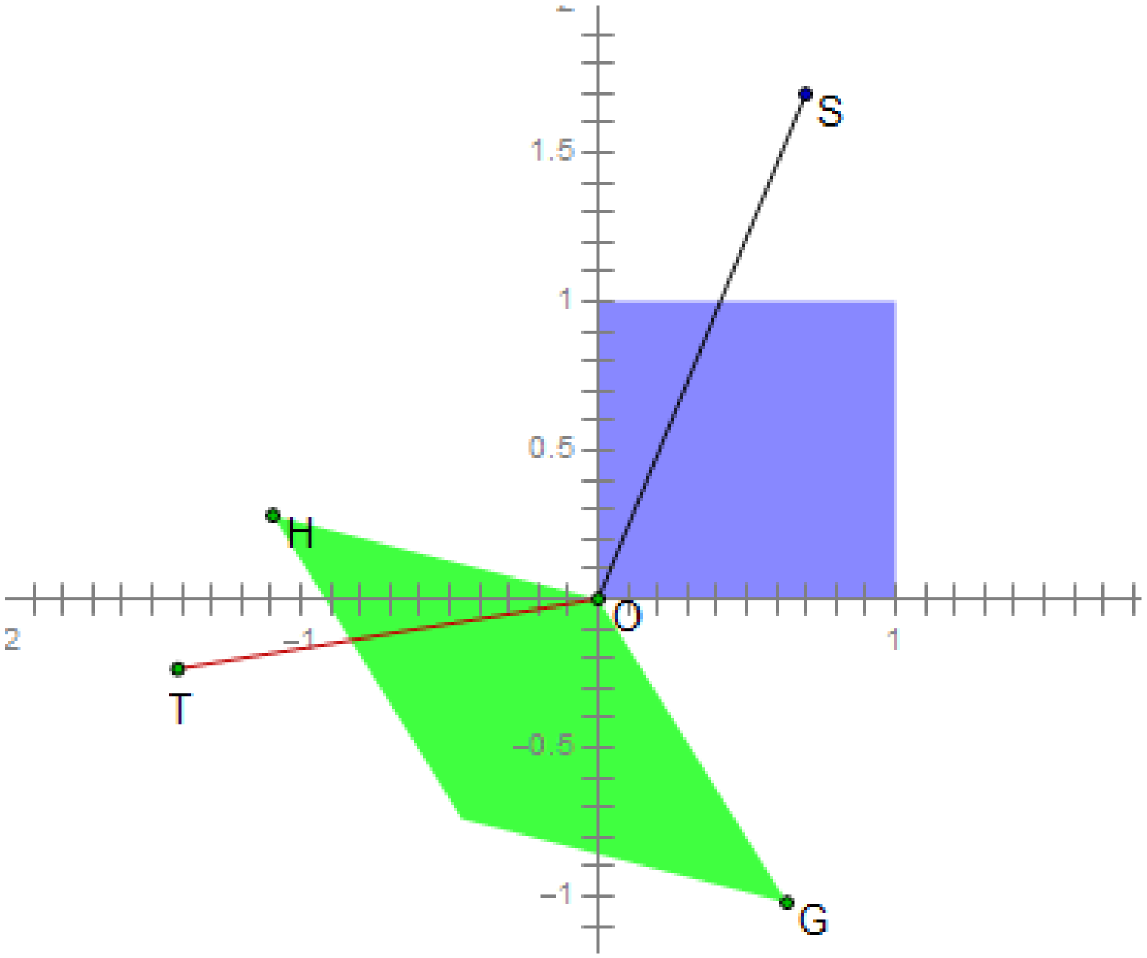
Geometric Interpretation of Eigenvalues in R^2

Suppose

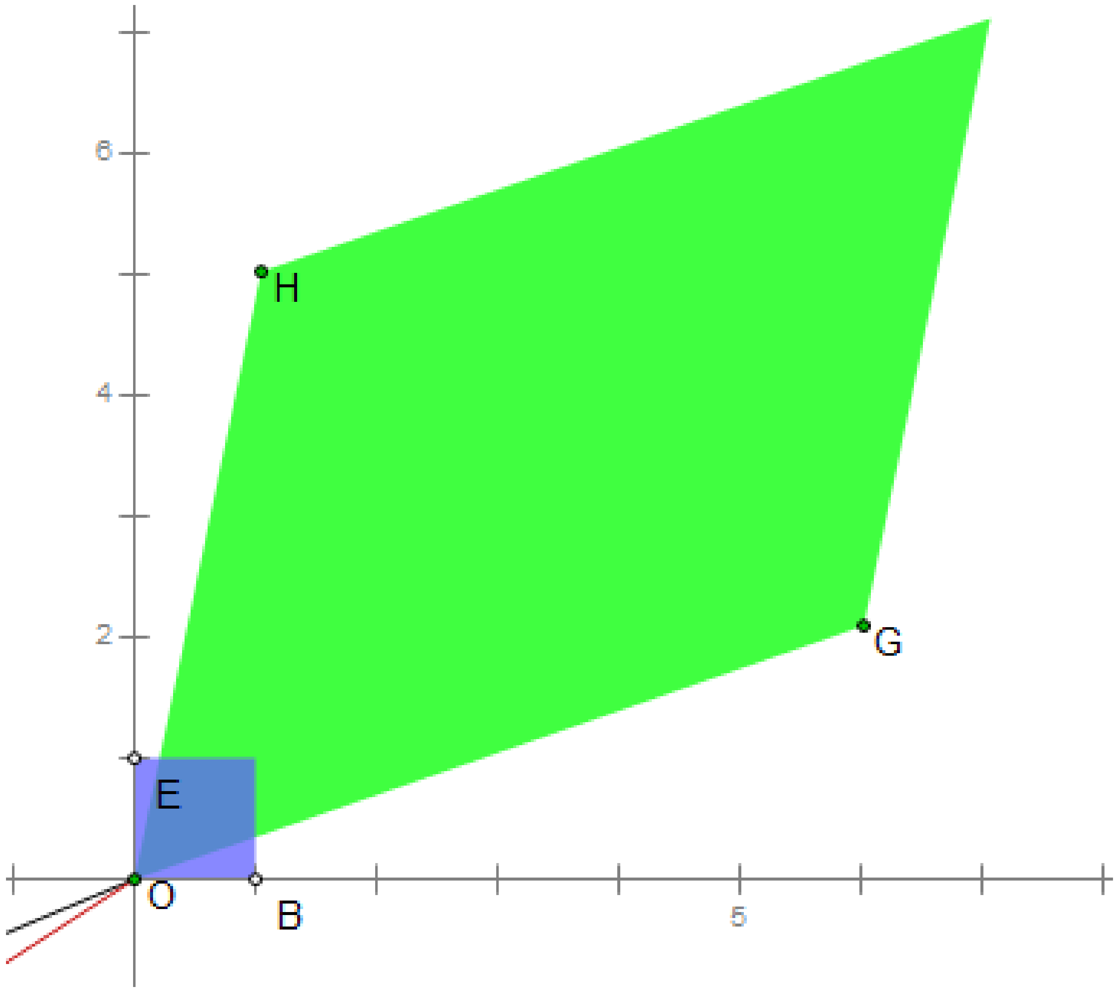
$$M = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}$$

It is easy to check that the eigenvalues of M are 7 and -4.

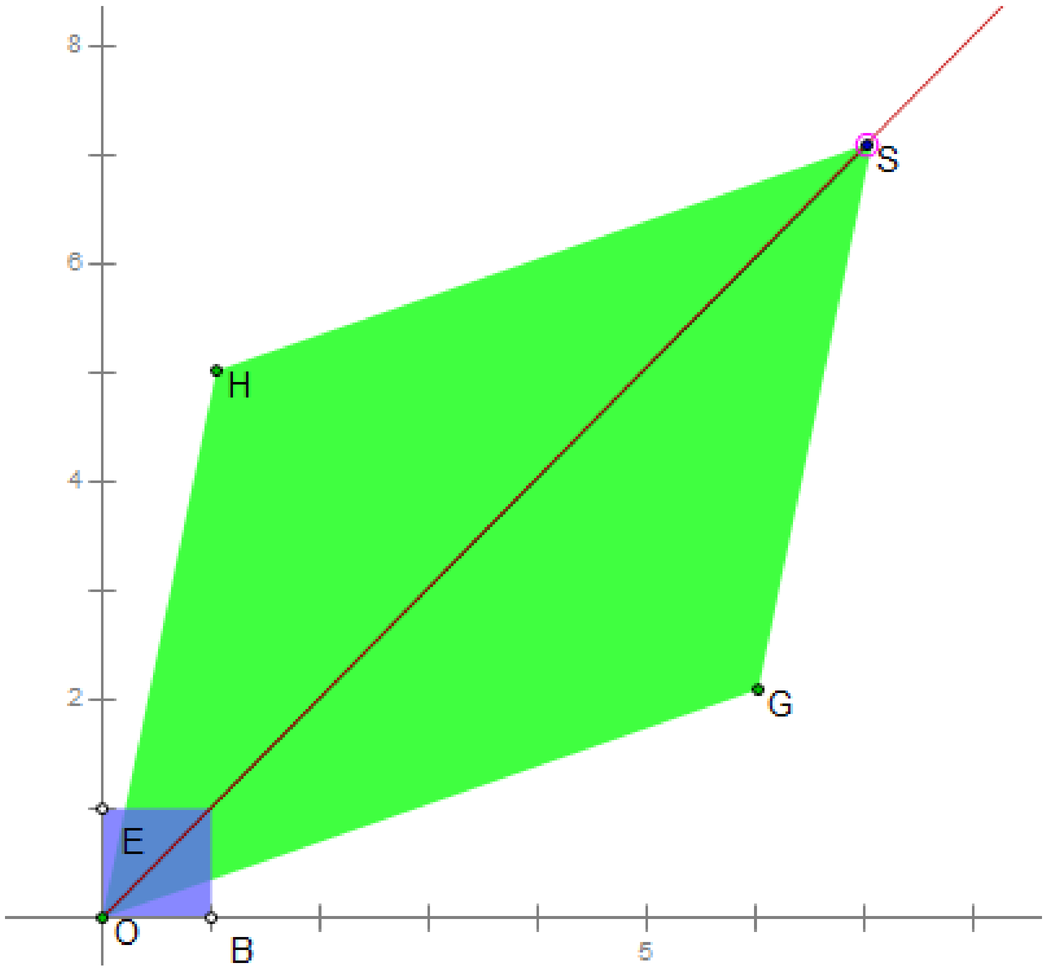
Geometer's Sketchpad shows Eigenvalues and Eigenvectors



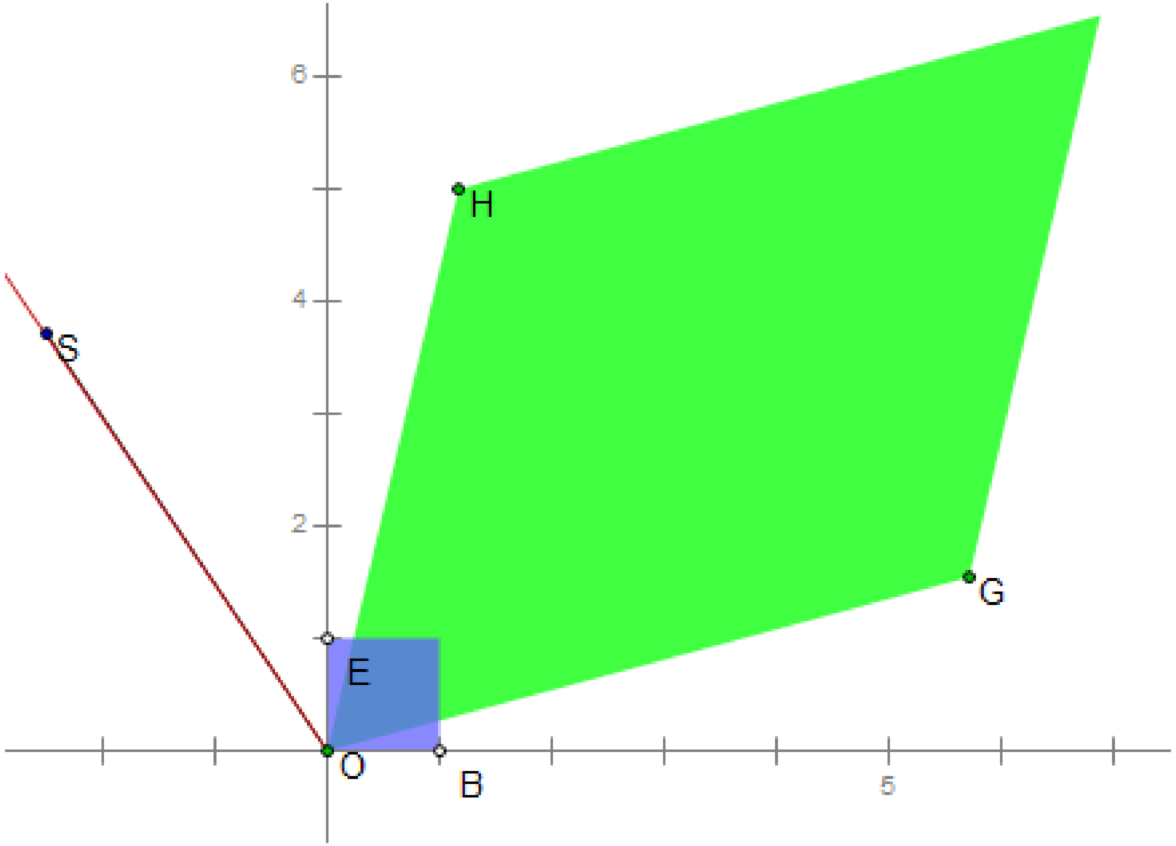
$$M = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}$$



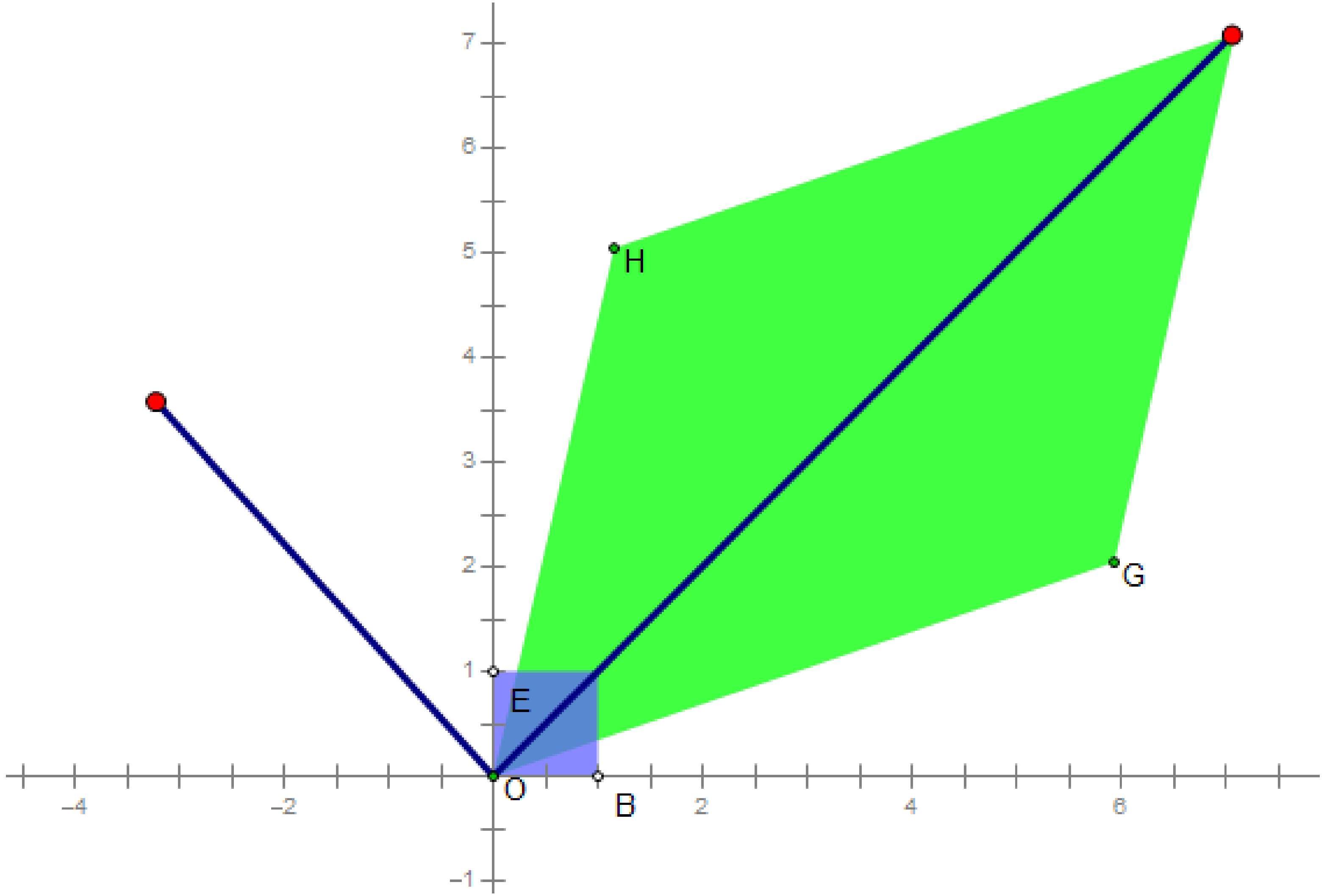
λ = 7



λ = -4



All together now...
the diagonals are the eigenvectors...right?

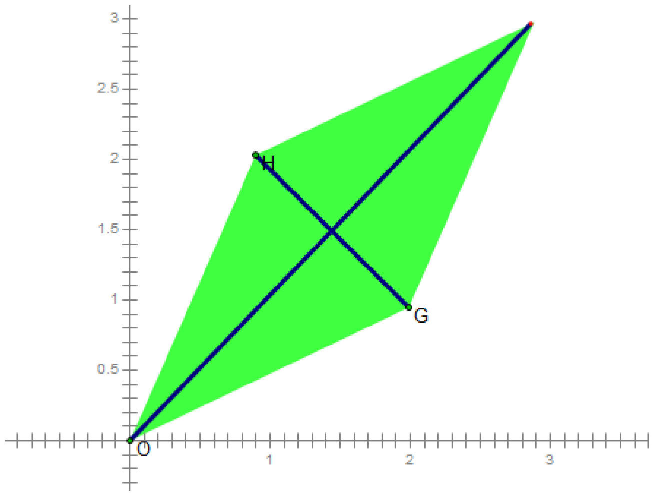


WRONG!!!

I want a 2 x 2 matrix [$\mathbf{v}_1$ $\mathbf{v}_2$] so that

(i) $\mathbf{v}_1 + \mathbf{v}_2$ is an eigenvector

(ii) $\mathbf{v}_1 - \mathbf{v}_2$ is an eigenvector



I will call these di-eigenals

What works? 2 x 2 eigenstuff….

- Let M=$\begin{bmatrix} a & b \\ b & a \end{bmatrix}$.

The eigenvalues of M are *a+b* and *a-b.*

The eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, respectively.

The diagonals of the image are $\begin{bmatrix} a+b \\ a+b \end{bmatrix} = (a+b)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} a-b \\ b-a \end{bmatrix} = (a-b)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Which is exactly what we wanted.

Doesn't Work → $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 & 6 \\ 6 & 1 \end{bmatrix}$ ← Works!

Proposition:

Let M be a 2x2 matrix. Then M has a full set of di-eigenals iff M is of the form $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ where $a+b \neq 0$ and $a-b \neq 0$

Proof: "If" follows immediately from the relevant definitions

"Only if" : Set up the required system of equations and let Mathematica do the work.

On beyond two dimensions….

In two dimensions, we could find the di-eigenals and saw that this happened when the image of the unit square is a rhombus

What about *n=3*?

Problem 1: What do you call it?

Parallelirhomboid

Rhombollelipiped

Rhombohedron

Problem 2: there are four diagonals and three eigenvectors.

$$4 \neq 3$$

Forget n=3. Try n=4.

We want a 4x4 matrix whose eigenvectors are some four vectors in R^4 that correspond to the diagonals of the image of the unit square under that action of the matrix.

What worked before? $M=\begin{bmatrix} a & b \\ b & a \end{bmatrix}$.

The sum of the columns gives one eigenvalue and the difference gives the other.

That is, the eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and the corresponding eigenvalues are $a+b = (a, b) \cdot (1,1)$ and $a-b = (a - b) \cdot (1,-1)$

Generalize….

IDEA: Label each “vertex” as a string of 1s and 0s.

Diagonals are differences of the vertices, so they are strings of 0s, 1s and -1s.

Diagonals of interest (i.e., not edges or diagonals of faces) are those whose strings have no 0s.

Example:

$(1,1,1,1) - (0,0,0,0) = (1,1,1,1)$

$(1,0,1,1) - (0,1,0,0) = (1, -1, 1, 1)$

These have potential

$(1,0,1,1) - (0,1,1,1) = (1, -1, 0, 0)$

For n = 4, there are 16 vertices and 8 potential diagonals

(we consider (1, -1,1, -1) equivalent to its negative, (-1, 1, -1 ,1).

But we can't have any more than 4 di-eigenals.

How to choose?

Look for some structure…

Think of the vertices as elements of Z_2 x Z_2x Z_2 x Z_2

Let H be the subgroup:

(1,1,1,1) (1,1,0,0) (0,1,1,0) (1,0,1,0)

(0,0,0,0) (0,0,1,1) (1,0,0,1) (0,1,0,1)

Note that K= {(1,1,1,1),(0,0,0,0)} is a subgroup of H.

Choose our four diagonals by pairing up the elements of the cosets of K in H and taking the differences. This gives

(1,1,1,1), (1,1,-1,-1), (1, -1, -1, 1) and (1, -1, -1, 1)

Are our diagonals actually di-eigenals?

YES!! Reverse engineering --

Choose the vectors

$$(1,1,1,1),\ (1,1,-1,-1),\ (1,-1,-1,1) \text{ and } (1,-1,1,-1)$$

to be eigenvectors and the corresponding dot products

$$a+b+c+d,\ \ a+b-c-d,\ a-b-c+d, \text{ and } a-b+c-d$$

to be the corresponding eigenvalues.

Let P be the matrix whose columns are these eigenvectors and D the diagonal matrix formed from the eigenvalues.

Let $M = PDP^{-1}$. And we're (almost) done!

And here it is: $M= \begin{matrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{matrix}$

We choose a, b, c, and d so that the eigenvalues are distinct and non-zero. This boils down to:

$b \pm d \neq 0$	$a+b+c+d \neq 0$
$b \pm c \neq 0$	$a+b-(c+d) \neq 0$
$c \pm d \neq 0$	$a+d-(b+c) \neq 0$
	$a+c-(b+d) \neq 0$

The same plan works for n = 8…

$$\begin{pmatrix} a & b & c & d & f & g & h & j \\ b & a & d & c & g & f & j & h \\ c & d & a & b & h & j & f & g \\ d & c & b & a & j & h & g & f \\ f & g & h & j & a & b & c & d \\ g & f & j & h & b & a & d & c \\ h & j & f & g & c & d & a & b \\ j & h & g & f & d & c & b & a \end{pmatrix}$$

A pattern emerges....

- N= 2

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

N=4

$$\begin{pmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{pmatrix}$$

N=8

$$\begin{pmatrix} a & b & c & d & f & g & h & j \\ b & a & d & c & g & f & j & h \\ c & d & a & b & h & j & f & g \\ d & c & b & a & j & h & g & f \\ f & g & h & j & a & b & c & d \\ g & f & j & h & b & a & d & c \\ h & j & f & g & c & d & a & b \\ j & h & g & f & d & c & b & a \end{pmatrix}$$

General plan of attack….

- Find a full set of linearly independent vectors of 1s and -1s. Make them the columns of a matrix P. Take the dot product of each vector with $(x_1,x_2,\ldots)$ to get the corresponding eigenvalues and use them to make a diagonal matrix D. Compute $M = PDP^{-1}$.

- Choose $(x_1,x_2,\ldots)$ (the entries of M) so that the eigenvalues are distinct and non-zero.

Problem: How to choose that set of linearly independent vectors????

IDEA: The plan….

- Let B be the 2 x 2 matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
- Create a 4 x 4 matrix P of eigenvectors by replicating B as follows:

$$\left(\begin{array}{c|c} B & B \\ \hline B & -B \end{array}\right) \longrightarrow \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{1} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{-1} & \mathbf{1} \end{pmatrix}$$

Iterate!! (These are better known as Hadamard matrices.)

Back to n = 3…

After some experimentation, it turns out that any set of three vectors of 1s and -1s produces di-eigenals.

Set P = $\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$ and eigenvalues {a+b+c,-a+b-c,-a-b+c} to produce

the matrix $\begin{pmatrix} -a & a+c & a+b \\ -b+c & b & a+b \\ b-c & a+c & c \end{pmatrix}$ which has three di-eigenals.

Moreover, the same Hadamard trick produces di-eigenals for n = 6. ETC.

Questions to ponder….

1. By design, the Hadamard plan works, but are there other ways to make di-eigenals?

2. The eigenvectors for the case $n = 2^k$ are mutually orthogonal.
   Is that necessary?

3. Is there some simple way to describe the sets of matrix entries that give non-zero distinct eigenvalues? (assuming solutions exist…) If so, is there some interesting geometry associated with this complement of a union of hyperplanes?

Thanks!	Thanks!
Thanks!	-Thanks!