Linear Algebra using Curated Courses Open Educational Resources

Sarah Eichhorn, *University of Texas at Austin*
Petra Bonfert-Taylor, *Dartmouth College*
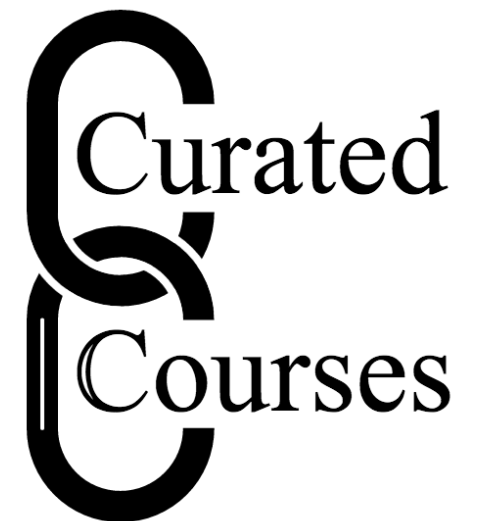**David Farmer,** *American Institute of Mathematics*
Jim Fowler, *The Ohio State University*

Joint Mathematics Meeting
January 11, 2018

Curated Courses

https://curatedcourses.org/

Scenario 1:

Thu 1/4/2018 11:35 AM

Eichhorn, Sarah E <s.eichhorn@aus

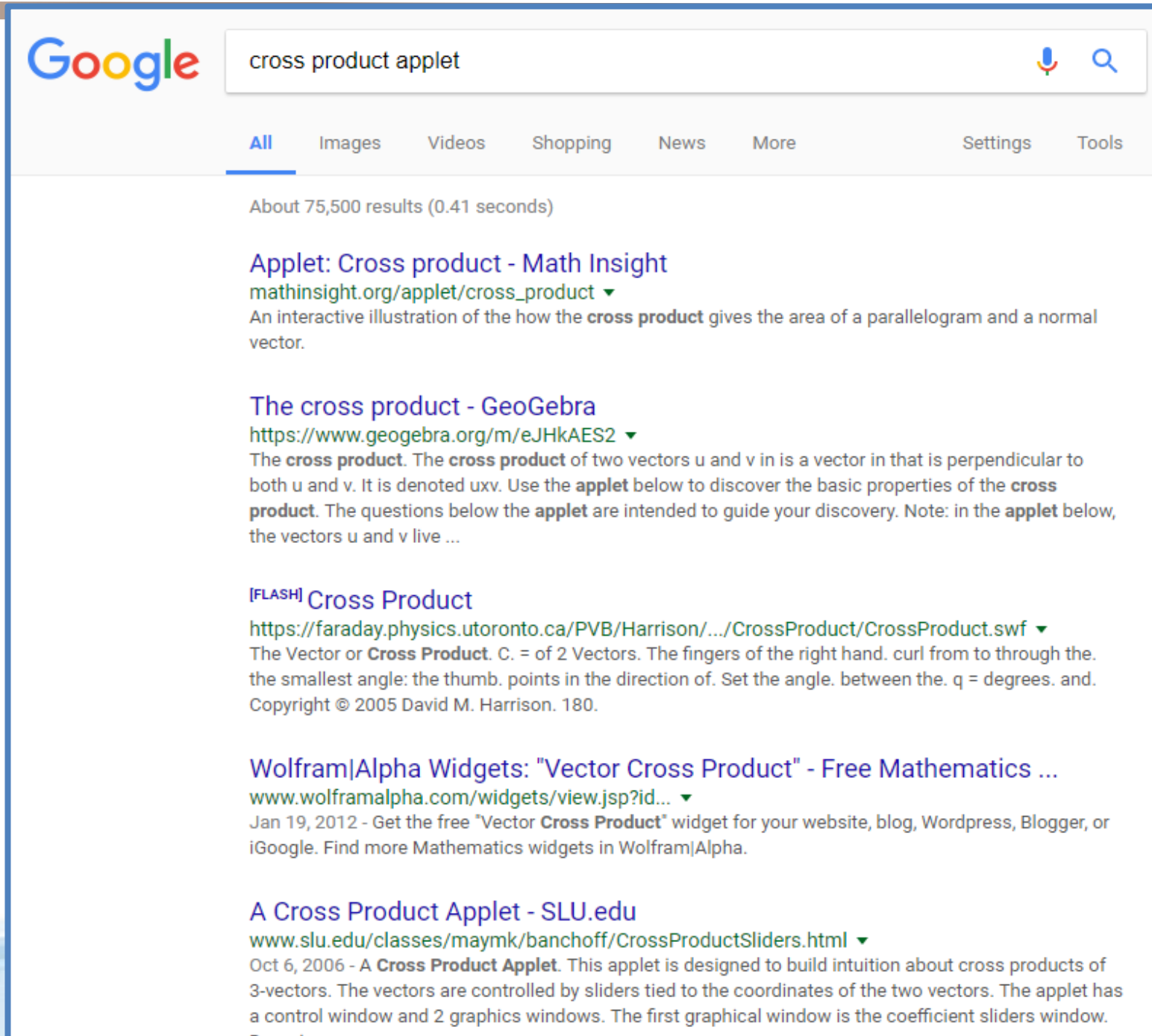**missed class today**

To   s.eichhorn@utexas.edu

Hey Dr. E,

I missed class today. Did you do anything important?

Could you give me a mini-lecture on what I missed?

-Joe Student

Scenario 2:

Outline

- What is Curated Courses?

- How can Curated Courses improve Linear Algebra instruction?

- What have we learned about making Linear Algebra content usable in variety of instructional settings?

What is Curated Courses?

- Project to **<u>create</u>** and **<u>curate</u>** high quality online open educational mathematics resources.

- Curate = gather, tag, organize, review, make available

What is Curated Courses?

- Website: https://curatedcourses.org

- Curated Courses currently contains resources for **Linear Algebra**

- Types of resources available:
  - Videos
  - Worksheets
  - Quizzes
  - Lesson plans
  - Clicker questions
  - Applets
  - Etc.

What is Curated Courses?

What is Curated Courses?

Linear Algebra and Its Applications
by David C. Lay, Steven R. Lay, and Judi J. McDonald

Linear Algebra and It
by David C. Lay, Steven R. Lay, and Judi J. McDonald

1 Linear Equations in Linear Algebra

2 Matrix Algebra

3 Determinants

4 Vector Spaces

5 Eigenvalues and Eigenvectors

6 Orthogonality and Least Squares

Linear Algebra and
by David C. Lay, Steven R. Lay, and Judi J. McDonald

1 Linear Equations in Linear Algebra

1.1 Systems of Linear Equations

1.2 Row Reduction and Echelon Forms

1.3 Vector Equations

1.4 The Matrix Equation Ax = b

1.5 Solution Sets of Linear Systems

1.6 Applications of Linear Systems

1.7 Linear Independence

1.8 Introduction to Linear Transformations

1.9 The Matrix of a Linear Transformation

1.10 Linear Models in Business, Science, and En

Supplementary Exercises

2 Matrix Algebra

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1 Linear Equations in Linear Algebra

1.1 Systems of Linear Equations

rref-augmented.sage

Sage cell illustrating creating a coefficient matrix of a system of three equations in three variables, augmenting with a vector of constants, and bringing the matrix to reduced row-echelon form in order to find the (unique) solution.

Linear systems have zero, one, or infinitely many solutions.
math.la.t.linsys.zoi

License	GFDL-1.3
Created On	December 30th, 2016
	a year ago
View	1
Type	SageMath Cell
Timeframe	Post-class
Perspective	Example
Language	English
Content Type	application/octet-stream

When does a linear system have a unique solution?

A 3x3 system having a unique solution is solved by putting the augmented matrix in reduced row echelon form. A picture of three intersecting planes provides geometric intuition.

Example of solving a 3-by-3 system of linear equations by row-reducing the augmented matrix, in the case of one solution
math.la.a.linsys.3x3.soln.row_reduce.o

Definition of matrix in reduced row echelon form math.la.d.mat.rref

Linear systems have zero, one, or infinitely many solutions.
math.la.t.linsys.zoi

Definition of consistent linear system math.la.d.linsys.consistent

Created On	February 15th, 2017
	a year ago
View	1
Type	Video
Timeframe	Review
Perspective	Example
Language	English
Content Type	text/html; charset=utf-8

Matrix equations with zero or infinitely many solutions

A 3x3 matrix equation Ax=b is solved for two different values of b. In one case there is no solution, and in another there are infinitely many solutions. These examples illustrate a theorem about linear combinations of the columns of the matrix A.

Example of solving a 3-by-3 system of linear equations by row-reducing the augmented matrix, in the case of no solutions
math.la.a.linsys.3x3.soln.row_reduce.z

Example of solving a 3-by-3 system of linear equations by row-reducing the augmented matrix, in the case of infinitely many solutions math.la.a.linsys.3x3.soln.row_reduce.i

The matrix equation Ax=b has a solution if and only if b is a linear combination of the columns of A. math.la.t.mat.eqn.lincomb

Created On	February 15th, 2017
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Perspective	Example
Language	English
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SolutionSetsInconsistent

$$\begin{bmatrix} 1 & -1 & 0 & | & 1 \\ 0 & 1 & 3 & | & 2 \\ 1 & 1 & 6 & | & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & | & 1 \\ 0 & 1 & 3 & | & 2 \\ 0 & 2 & 6 & | & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & | & 1 \\ 0 & 1 & 3 & | & 2 \\ 0 & 0 & 0 & | & -3 \end{bmatrix}$$

Pen selected

$\mathcal{CC}i\mathbf{M}$

The reduced row echelon form is used to determine when a 3x3 system is inconsistent. A picture of planes in 3-dimensional space is used to provide geometric intuition.

Created On	February 15th, 2017
	a year ago
View	1
Type	◼ Video
Timeframe	↻ Review
Perspective	↟ Example
Language	▇ English
Content Type	text/html; charset=utf-8

Example of solving a 3-by-3 system of linear equations by row-reducing the augmented matrix, in the case of no solutions math.la.e.linsys.3x3.soln.row_reduce.z

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Improving Instruction

For students…

- Alternative explanations of concepts

- Additional worked examples

- Textbook aligned materials for exam preparation review

- Interactive applets and visualizations

Improving Instruction

For faculty…

- Flipped or online course materials

- In-class activities

- Pre-class preparation

- Post-class review

- Interactive applet activities

- Make-up lectures

- Review materials for other classes relying on Linear Algebra content

Content Versatility

- Adopt consistent, common notation scheme

- Avoid numbering (theorems, definitions, equations, etc.)

- Identify required prerequisite content

- Review previous content as needed

- Provide source files

- Use standard formats

- Modularize

- Use open licensing
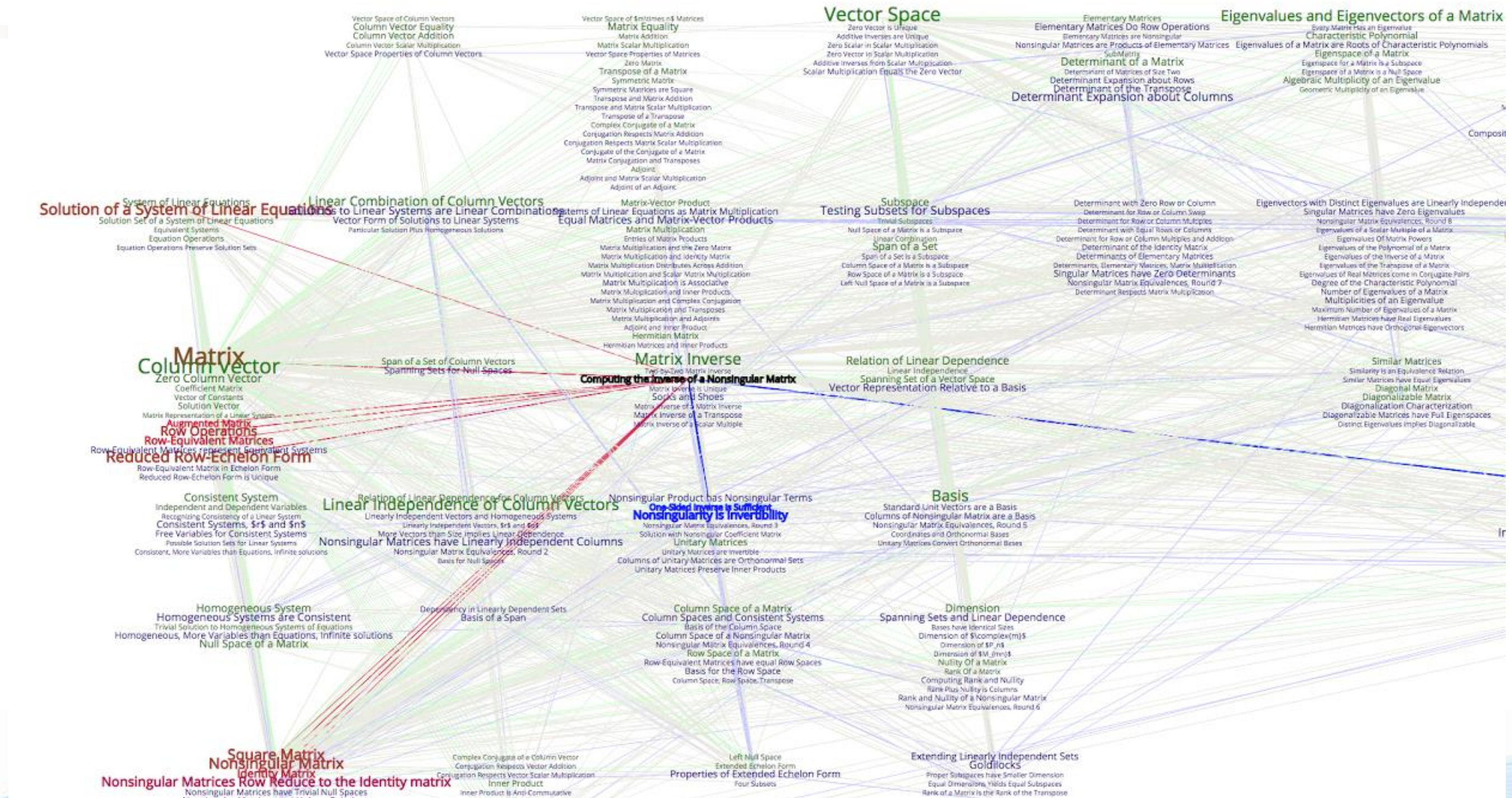
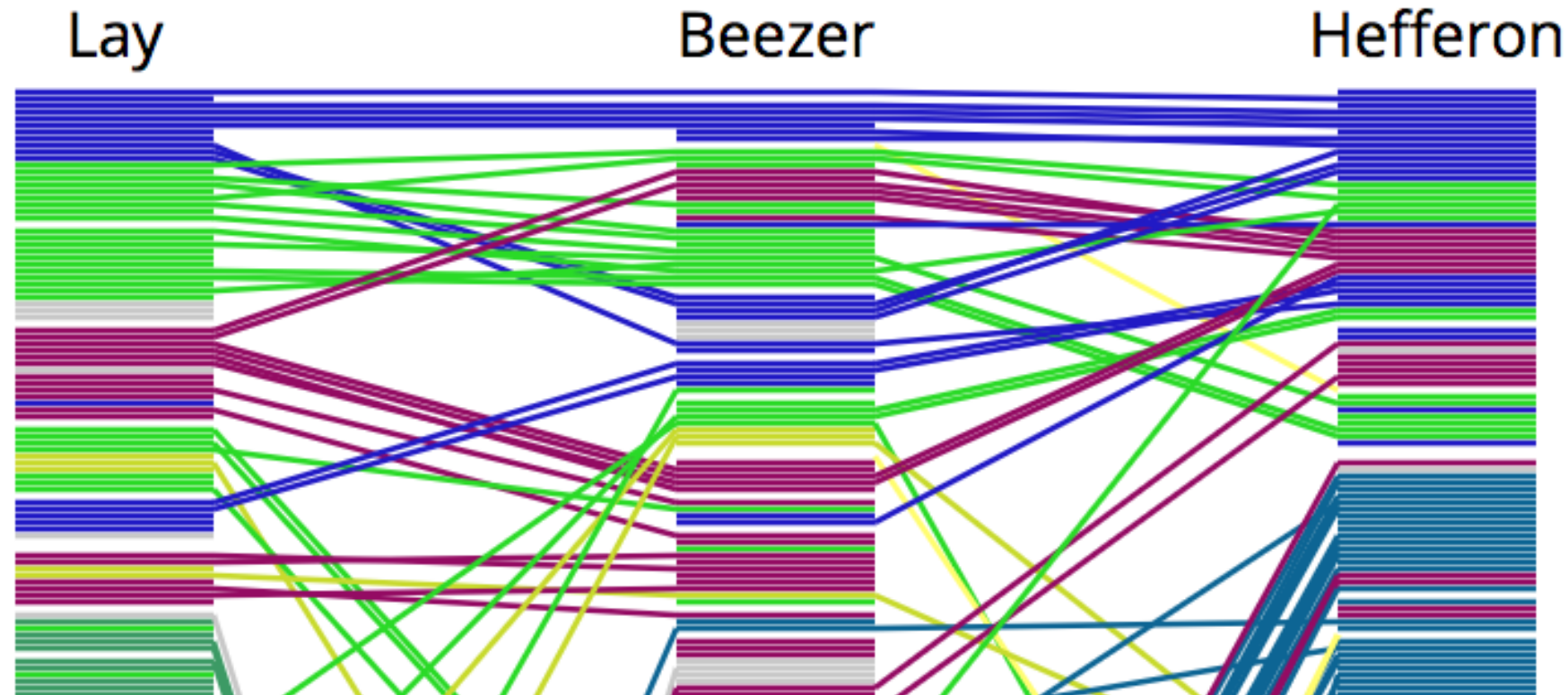# Systems of Linear Equations   Vectors   Matrices   Vector Spaces   Determinants   Eigenvalues   Lir

Vector Space

Vector Space of Column Vectors
Column Vector Equality
Column Vector Addition
Column Vector Scalar Multiplication
Vector Space Properties of Column Vectors

Vector Space of $m\times n$ Matrices
Matrix Equality
Matrix Addition
Matrix Scalar Multiplication
Vector Space Properties of Matrices
Zero Matrix
Transpose of a Matrix
Symmetric Matrix
Symmetric Matrices are Square
Transpose and Matrix Addition
Transpose and Matrix Scalar Multiplication
Transpose of a Transpose
Complex Conjugate of a Matrix
Conjugation Respects Matrix Addition
Conjugation Respects Matrix Scalar Multiplication
Conjugate of the Conjugate of a Matrix
Matrix Conjugation and Transposes
Adjoint
Adjoint and Matrix Scalar Multiplication
Adjoint of an Adjoint

Zero Vector is Unique
Additive Inverses are Unique
Zero Scalar in Scalar Multiplication
Zero Vector in Scalar Multiplication
Additive Inverses from Scalar Multiplication
Scalar Multiplication Equals the Zero Vector

Elementary Matrices
Elementary Matrices Do Row Operations
Elementary Matrices are Nonsingular
Nonsingular Matrices are Products of Elementary Matrices
SubMatrix
Determinant of a Matrix
Determinant of Matrices of Size Two
Determinant Expansion about Rows
Determinant of the Transpose
Determinant Expansion about Columns

Eigenvalues and Eigenvectors of a Matrix
Every Matrix Has an Eigenvalue
Characteristic Polynomial
Eigenvalues of a Matrix are Roots of Characteristic Polynomials
Eigenspace of a Matrix
Eigenspace for a Matrix is a Subspace
Eigenspace of a Matrix is a Null Space
Algebraic Multiplicity of an Eigenvalue
Geometric Multiplicity of an Eigenvalue

Composi

Solution of a System of Linear Equations
System of Linear Equations
Solution Set of a System of Linear Equations
Equivalent Systems
Equation Operations
Equation Operations Preserve Solution Sets

Linear Combination of Column Vectors
Solutions to Linear Systems are Linear Combinations
Vector Form of Solutions to Linear Systems
Particular Solution Plus Homogeneous Solutions

Matrix-Vector Product
Systems of Linear Equations as Matrix Multiplication
Equal Matrices and Matrix-Vector Products
Matrix Multiplication
Entries of Matrix Products
Matrix Multiplication and the Zero Matrix
Matrix Multiplication and Identity Matrix
Matrix Multiplication Distributes Across Addition
Matrix Multiplication and Scalar Matrix Multiplication
Matrix Multiplication is Associative
Matrix Multiplication and Inner Products
Matrix Multiplication and Complex Conjugation
Matrix Multiplication and Transposes
Matrix Multiplication and Adjoints
Adjoint and Inner Product
Hermitian Matrix
Hermitian Matrices and Inner Products

Subspace
Testing Subsets for Subspaces
Trivial Subspaces
Null Space of a Matrix is a Subspace
Linear Combination
Span of a Set
Span of a Set is a Subspace
Column Space of a Matrix is a Subspace
Row Space of a Matrix is a Subspace
Left Null Space of a Matrix is a Subspace

Determinant with Zero Row or Column
Determinant for Row or Column Swap
Determinant for Row or Column Multiples
Determinant with Equal Rows or Columns
Determinant for Row or Column Multiples and Addition
Determinant of the Identity Matrix
Determinants of Elementary Matrices
Determinant, Elementary Matrices, Matrix Multiplication
Singular Matrices have Zero Determinants
Nonsingular Matrix Equivalences, Round 7
Determinant Respects Matrix Multiplication

Eigenvectors with Distinct Eigenvalues are Linearly Independent
Singular Matrices have Zero Eigenvalues
Nonsingular Matrix Equivalences, Round 8
Eigenvalues of a Scalar Multiple of a Matrix
Eigenvalues Of Matrix Powers
Eigenvalues of the Polynomial of a Matrix
Eigenvalues of the Inverse of a Matrix
Eigenvalues of the Transpose of a Matrix
Eigenvalues of Real Matrices come in Conjugate Pairs
Degree of the Characteristic Polynomial
Number of Eigenvalues of a Matrix
Multiplicities of an Eigenvalue
Maximum Number of Eigenvalues of a Matrix
Hermitian Matrices have Real Eigenvalues
Hermitian Matrices have Orthogonal Eigenvectors

Matrix
Column Vector
Zero Column Vector
Coefficient Matrix
Vector of Constants
Solution Vector
Matrix Representation of a Linear System
Augmented Matrix
Row Operations
Row-Equivalent Matrices
Row-Equivalent Matrices represent Equivalent Systems
Reduced Row-Echelon Form
Row-Equivalent Matrix in Echelon Form
Reduced Row-Echelon Form is Unique

Span of a Set of Column Vectors
Spanning Sets for Null Spaces

Matrix Inverse
Two-by-Two Matrix Inverse
Computing the Inverse of a Nonsingular Matrix
Matrix Inverse is Unique
Socks and Shoes
Matrix Inverse of a Matrix Inverse
Matrix Inverse of a Transpose
Matrix Inverse of a Scalar Multiple

Relation of Linear Dependence
Linear Independence
Spanning Set of a Vector Space
Vector Representation Relative to a Basis

Similar Matrices
Similarity is an Equivalence Relation
Similar Matrices have Equal Eigenvalues
Diagonal Matrix
Diagonalizable Matrix
Diagonalization Characterization
Diagonalizable Matrices have Full Eigenspaces
Distinct Eigenvalues implies Diagonalizable

Consistent System
Independent and Dependent Variables
Recognizing Consistency of a Linear System
Consistent Systems, r and n
Free Variables for Consistent Systems
Possible Solution Sets for Linear Systems
Consistent, More Variables than Equations, Infinite solutions

Relation of Linear Dependence for Column Vectors
Linear Independence of Column Vectors
Linearly Independent Vectors and Homogeneous Systems
Linearly Independent Vectors, r and n
More Vectors than Size implies Linear Dependence
Nonsingular Matrices have Linearly Independent Columns
Nonsingular Matrix Equivalences, Round 2
Basis for Null Space

Nonsingular Product has Nonsingular Terms
One-Sided Inverse is Sufficient
Nonsingularity is Invertibility
Nonsingular Matrix Equivalences, Round 3
Solution with Nonsingular Coefficient Matrix
Unitary Matrices
Unitary Matrices are invertible
Columns of Unitary Matrices are Orthonormal Sets
Unitary Matrices Preserve Inner Products

Basis
Standard Unit Vectors are a Basis
Columns of Nonsingular Matrix are a Basis
Nonsingular Matrix Equivalences, Round 5
Coordinates and Orthonormal Bases
Unitary Matrices Convert Orthonormal Bases

In

Homogeneous System
Homogeneous Systems are Consistent
Trivial Solution to Homogeneous Systems of Equations
Homogeneous, More Variables than Equations, Infinite solutions
Null Space of a Matrix

Dependency in Linearly Dependent Sets
Basis of a Span

Column Space of a Matrix
Column Spaces and Consistent Systems
Basis of the Column Space
Column Space of a Nonsingular Matrix
Nonsingular Matrix Equivalences, Round 4
Row Space of a Matrix
Row-Equivalent Matrices have equal Row Spaces
Basis for the Row Space
Column Space, Row Space, Transpose

Dimension
Spanning Sets and Linear Dependence
Bases have Identical Sizes
Dimension of $\complex{m}$
Dimension of P_n
Dimension of M_{mn}
Nullity Of a Matrix
Rank Of a Matrix
Computing Rank and Nullity
Rank Plus Nullity is Columns
Rank and Nullity of a Nonsingular Matrix
Nonsingular Matrix Equivalences, Round 6

Square Matrix
Nonsingular Matrix
Identity Matrix
Nonsingular Matrices Row Reduce to the Identity matrix
Nonsingular Matrices have Trivial Null Spaces
Nonsingular Matrices and Unique Solutions

Complex Conjugate of a Column Vector
Conjugation Respects Vector Addition
Conjugation Respects Vector Scalar Multiplication
Inner Product
Inner Product is Anti-Commutative
Norm of a Vector

Left Null Space
Extended Echelon Form
Properties of Extended Echelon Form
Four Subsets

Extending Linearly Independent Sets
Goldilocks
Proper Subspaces have Smaller Dimension
Equal Dimensions Yields Equal Subspaces
Rank of a Matrix is the Rank of the Transpose

Content Versatility

Legend:
- Systems of linear equations
- Matrices
- Determinants
- Equivalences to 'nonsingular'
- Vectors
- Vector spaces
- Linear transformations

Lay — Beezer — Hefferon

Getting Involved

Curated Courses welcomes…

- Uploads of Linear Algebra content to curatedcourses.org

- Feedback from content users via curatedcourses.org/contact

- Volunteer content reviewers

- Developers to help expand Curated Courses beyond Linear Algebra (next targets: Differential Equations and Abstract Algebra)

- Contact: Sarah Eichhorn (s.eichhorn@austin.utexas.edu)

- Thank you!