Active Learning in Linear Algebra

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Overview

- Preview Activities
- In-class Activities
- Student Reaction & Evaluation

How do preview activities work?

- Assigned at the end of one class period – due the next class period.
- Material ties to the topic of the day and the in-class activities.
- Problem types:
  - Working with examples/counterexamples, or creating from basic definitions
  - Making connections between old topics to the new material
- Discussion of preview in the classroom/small group and whole class.

Vector Representation Preview Activity

Preview Activity 1.

(1) Determine the components of the vector $3\mathbf{v} + \mathbf{u} - 2\mathbf{w}$ where

$$\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}.$$

(2) In mathematics, any time we define operations on objects, such as addition of vectors, we ask which properties the operation has. For example, one might wonder if $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for any two vectors $\mathbf{u}, \mathbf{v}$ of the same size. If this property holds, we say that the *addition of vectors is a commutative operation*. However, to verify this property we cannot use examples since the property must hold for any two vectors. For simplicity, we focus on two-dimensional vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Using these arbitrary vectors, can we say that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$? If so, justify. If not, give a counterexample. (Note: Giving a counterexample is the best way to justify why a general statement is not true.)

Vector Representation Preview Activity –Part 2

(3) We can geometrically represent vectors with two components as points in the plane. Specifically, the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ corresponds to the point (x, y) in the plane. This representation will be especially handy when we consider infinite collections of vectors as we do in this problem.

(a) On the same set of axes, plot the points that correspond to 5-6 scalar multiples of the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Make sure to use variety of scalar multiples covering possibilities with $c > 0, c < 0, c > 1, 0 < c < 1, -1 < c < 0$. If we consider the collection of all possible scalar multiples of this vector, what do we obtain?

(b) What would the collection of all scalar multiples of the vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ form in the plane?

(c) What would the collection of all scalar multiples of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ form in the three-dimensional space?

(4) Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ in $\mathbb{R}^2$. We are interested in finding all vectors that can be formed as a sum of scalar multiples of $\mathbf{u}$ and $\mathbf{v}$.

(a) On the same set of axes, plot the points that correspond to the vectors $\mathbf{u}, \mathbf{v}, \mathbf{u}+\mathbf{v}, 1.5\mathbf{u}, 2\mathbf{v}, -\mathbf{u}, -\mathbf{v}, -\mathbf{u}+2\mathbf{v}$. Plot other random sums of scalar multiples of $\mathbf{u}$ and $\mathbf{v}$ using several scalar multiples (including those less than 1 or negative) (that is, find other vectors of the form $a\mathbf{u} + b\mathbf{v}$ where a and b are any scalars.).

(b) If we considered sums of all scalar multiples of $\mathbf{u}, \mathbf{v}$, which vectors will we obtain? Can we obtain any vector in $\mathbb{R}^2$ in this form?

Why do we use preview activities?

- Save class time.
- Students come prepared having thought about the new material – are prepared for class work and discussion.
- Practice for mathematical skills including conjecturing, recognizing patterns and using appropriate tools.
- Build new knowledge on existing knowledge.
- Helps students learn to read mathematics and to apply theorems and definitions.
- Helps students take more responsibility in their learning.

Vector Representation In-Class Activity

In the activity below we consider how the two operations, addition and scalar multiplication, interact with each other. In real numbers, we know that multiplication is distributive over addition. Is that true with vectors as well?

Activity 1. We work with vectors in $\mathbb{R}^2$ to make the notation easier.

Let a be an arbitrary scalar, and $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be two *arbitrary* vectors in $\mathbb{R}^2$. Is $a(\mathbf{u}+\mathbf{v})$ equal to $a\mathbf{u}+a\mathbf{v}$? What property does this imply about the scalar multiplication and addition operations on vectors?

Vector Representation In-Class Activity –Part 2

Activity 2. Our chemical solution example illustrates that we might want to determine whether certain vectors can be written as a linear combination of given vectors. We explore that idea in more depth in this activity. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.

(a) Calculate the linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ with corresponding weights (scalar multiples) 1 and 2. The resulting vector is a vector which can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

(b) Can $\mathbf{w} = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$ be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$? If so, which linear combination? If not, explain why not?

Vector Representation In-Class Activity –Part 3

(c) Can $\mathbf{w} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$ be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$? If so, which linear combination? If not, explain why not?

(d) Let $\mathbf{w} = \begin{bmatrix} 0 \\ 6 \\ -2 \end{bmatrix}$. The problem of determining if $\mathbf{w}$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ is equivalent to the problem of finding scalars x_1 and x_2 so that

$$\mathbf{w} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2. \tag{1}$$

i. Combine the vectors on the right hand side of equation (1) into one vector, and then set the components of the vectors on both sides equal to each other to convert the vector equation (1) to a linear system of three equations in two variables.

ii. Use row operations to find a solution, if it exists, to the system you found in the previous part of this activity. If you find a solution, verify in (1) that you have found appropriate weights to produce the vector $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Vector Representation In-Class Activities

- Other in-class activities for vector representation have students
  - represent a system as a vector equation;
  - find the span of a few collections of vectors and make conjectures on the geometric descriptions of the spans in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Representative student evaluations

- I personally liked/benefited from the self-paced discovery learning assignments, they helped me reach a greater understanding of topics by uncovering results on my own. Pre-class activities helped me stay engaged and are a great resource when studying for exams.
- The pre class activities were quite useful because you were introducing yourself to new material before being taught in class which helped prime the mind to absorb the information.
- I think the group aspect to the class helped my learning most. The Professor really encouraged us to talk to each other and participate in class. This helped me because I got to talk out my thinking with my peers and therefore I further learned the information.
- The in class activities ensured that if I did not understand material I had many opportunities to figure that out and to better understand the material.

Representative student evaluations

- My instructor could have gone over more examples in class instead of just having the students try to figure it out in pre-class activities, or give more clarification.
- Less class activities, more lecturing and explanation.
- I like it when you go over things on the board first, instead of discovery based learning.
- Pre-class activities might do better having a provided answer to check your own work, instead of being able to produce a false result with no easy way to check.
- I don't like how open the class is for a math class. Half the time students are talking about unrelated subjects after arriving at wrong conclusions to questions posed in class.

Conclusions

- Students practice actively reading and wrestling with mathematics.
- Activities help students learn how to dissect mathematical problems.
- Monitoring student groups helps the instructor know what students really understand.
- Activities improve student attendance and participation.

Grand Valley State University Texts

- *Active Calculus* by Matt Boelkins, with David Austin and Steven Schlicker
- *Active Multivariable Calculus* by Steven Schlicker, with David Austin and Matt Boelkins (to appear soon)
- *Mathematical Reasoning: Writing and Proof* by Ted Sundstrom
- *Trigonometry* by Ted Sundstrom and Steven Schlicker
- *Linear Algebra: A Guided Discovery Approach* by Feryal Alayont and Steven Schlicker (available for Fall 2017?)

See Open Textbooks at `http://scholarworks.gvsu.edu/`

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Thank you for listening!

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