# Student Mathematical Connections in an Introductory Linear Algebra Course Employing a Hybrid of Inquiry-Oriented Teaching and Traditional Lecture

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# Mathematical Connections in Linear Algebra

"One of the most appealing aspects of linear algebra – yet a serious source of difficulty for students – is the "endless" number of mathematical connections one can (must) create studying it" (Harel, 1997).

Logical implication connections

- "If the columns of a matrix *A* are linearly independent, then no column of *A* is a linear combination of the other columns"

# The Invertible Matrix Theorem (IMT)

Let $A$ be a square $n \times n$ matrix. Then the following statements are equivalent.

- $A$ is an invertible matrix.
- $A$ is row equivalent to the $n \times n$ identity matrix.
- $A$ has $n$ pivot positions.
- The equation $A\mathbf{x}=\mathbf{0}$ has only the trivial solution.
- The columns of $A$ form a linearly independent set.
- The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
- The equation $A\mathbf{x}=\mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $R^n$.
- The columns of $A$ span $R^n$.
- The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $R^n$ onto $R^n$.
- There is an $n \times n$ matrix $C$ such that $CA=I$.
- There is an $n \times n$ matrix $D$ such that $AD=I$.
- $A^T$ is an invertible matrix.
- The columns of $A$ form a basis of $R^n$.
- Col $A=R^n$
- dimCol $A=n$
- rank $A=n$
- Nul $A=\{\mathbf{0}\}$
- dimNul $A=0$
- The number 0 is *not* an eigenvalue of $A$.
- The determinant of $A$ is *not* zero.

# Two More Theorems of Logical Equivalence

**Theorem 1:** Let $A$ be an $m \times n$ matrix. Then the following statements are logically equivalent.

- The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^m$.
- $A$ has $m$ pivot positions; that is, $A$ has a pivot position in every row.
- Every vector $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of $A$.
- The columns of $A$ span $\mathbb{R}^m$.
- The linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

# Two More Theorems of Logical Equivalence

**Theorem 2:** Let $A$ be an $m \times n$ matrix. Then the following statements are logically equivalent.

- The equation $A\mathbf{x} = \mathbf{b}$ has at most one solution for each $\mathbf{b}$ in $\mathbb{R}^m$.
- For each $\mathbf{b}$ in $\mathbb{R}^m$, the linear system corresponding to $A\mathbf{x} = \mathbf{b}$ does not have a free variable; that is, the linear system only has basic variables.
- $A$ has $n$ pivot positions; that is, $A$ has a pivot position in every column.
- The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- The columns of $A$ form a linearly independent set.
- No column of $A$ is a linear combination of the other columns.
- The linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one.

# My Linear Algebra Class

2 credit course

Content

- Linear systems, vector algebra, span and linear independence, linear transformations, matrix algebra, invertibility, subspaces of $\mathbb{R}^n$, determinants, eigen theory, and orthogonality

Normally 80-120 students

Coordinated sections

- Same content, textbook (Lay), schedule, exam, grading, policies

# Inquiry-Oriented Teaching

**Mathematical inquiry** (Richards, 1991)

- Mathematical discussion
- Solving math problems
- Forming conjectures

**Inquiry-oriented teaching:** practice of creating opportunities for students to engage in mathematical inquiry

# Research Questions

What does it look like when a teacher attempts to incorporate inquiry-oriented teaching in an undergraduate introductory linear algebra class?

What mathematical connections do students appear to evoke within the context of an introductory linear algebra course that employs inquiry-oriented teaching?

# Classroom Context

Introductory linear algebra courses that I taught

- Action research
  - Classroom observations
  - Written reflections
- Summer ‘15 (pilot study), Fall ‘15, spring ‘16

Accommodations for research

- Forty students in fall, sixty in spring
- Some freedom with exams and schedules

# Results: Inquiry-Oriented Teaching

## Inquiry-oriented teaching and lecture

- Hybrid approach

## Inquiry-oriented teaching

- Student development of math
- Specific teaching goals (connections)

## Lecture

- Connect to formal math
- Concepts not central to understanding of subject

# Inquiry in My Hybrid Approach

Reserved inquiry for big ideas
- Span, linear independence, etc.

Inquiry-oriented activities
- Quick
- Problems
- Discussion
- Connections

# Example: Span

Without solving a linear system or using any elementary row operations, determine whether the following sets of vectors span the given space. For each set of vectors, formulate a conjecture about span based on that set.

Do the vectors $\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ span all of $\mathbb{R}^2$?

Do the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ span all of $\mathbb{R}^3$?

Do the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ span all of $\mathbb{R}^3$?

Do the vectors $\begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 9 \\ 0 \end{bmatrix}$ span all of $\mathbb{R}^3$?

Do the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ span all of $\mathbb{R}^2$?

# The Hybrid Approach

Three considerations

- Definition of inquiry, inquiry-oriented teaching
- Teaching goals
- Constraints

Inquiry-oriented teaching emerged through reflection on these considerations

# Questions?

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