# Specific Examples, Generic Elements and Size Tuning - Tools for Overcoming Student Roadblocks in Linear Algebra

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# My Background

- Fifteen years at PLU - a medium-sized, liberal arts university
  - 15 – 20 math-related majors per year, all take linear algebra as sophomores or juniors.
  - Plus some physics, CS and econ majors, also some math/stats minors
  - We offer a single, proof-based linear algebra course
  - Half of students take an Intro to Proofs course first
- Fifteen years at a medium-sized state university with a masters in math and a joint PhD in Computational Science
  - 15 - 20 math majors per year, most of whom took two semesters of proof-based linear algebra
  - We also offered a low-proof matrix theory course for math ed, and two graduate computational matrix theory courses
  - Half of students in first proof-based course took Discrete Math first
- Thirty years as a researcher in combinatorial matrix theory
  - (Thank you, Professor Hans Schneider)

# Student Problems in Proofs (1)

- Linear Algebra is the often the first math course in which **sets** play an **explicit** and fundamental role.
- Linear Algebra students typically struggle with writing proofs for set-based results.

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- Prove that vector addition is commutative in $\mathbb{R}^n$.
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- Putative Proof: "$(4,3)$ and $(5,7)$ are in $\mathbb{R}^n$, and $(4,3)+(5,7)=(9,10)=(5,7)+(4,3)$."
- A **specific example** is employed to prove the truth of a universal statement, and a proof for $\mathbb{R}^2$ dispatches $\mathbb{R}^n$ for all $n$. Note that specifying $\mathbb{R}^{47}$ rather than $\mathbb{R}^n$ blocks student from using $\mathbb{R}^2$. Since 47 is too large to explicitly write each entry, students are pushed towards focusing on a general entry.

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- In the first "proof", a specific example "verifies" a universal statement.
- In the second "proof", a generic example falsely contradicts a false universal statement. The student fails to see the need for a specific example.

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- "We want to show a property can fail for some elements in a set. What is a often good way to begin?"
- This (eventually) prompts students to volunteer, "Look for a specific element (counterexample) in the set."

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- Most students can generate generic elements and perform the matrix algebra to correctly show that the sum of two $3 \times 3$ upper triangular matrices is upper triangular.

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- Most students can generate the generic elements and perform the matrix algebra to correctly show that the product of two $3 \times 3$ upper triangular matrices is upper triangular.

# Four Key Strategies for Student Proof Success

1. Emphasize the role of **specific (fully specified) examples** as examples to highlight definitions, and, more importantly, as counterexamples to universal statements.
2. Emphasize what a **generic element** from a set is, how to write one or more generic elements from a set, and what role they play in proofs about sets.
3. Emphasize the **different and noninterchangeable roles** of specific examples and generic elements.
4. **Thoughtfully tune the sizes** of vectors and matrices in problems to focus students on the primary idea at hand.

# Spans are the Unsung Heroes of Linear Algebra ... but ...

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- Size tuning suggests a cascade of examples and problems.

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- If $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k \in \mathbb{R}^n$, then $span(\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k) = \left\{\sum_{j=1}^k c_j\mathbf{v}_j : \text{all } c_j \in \mathbb{R}\right\}$.
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- If $S$ is a nonempty, possibly infinite subset of $\mathbb{R}^n$, then $span(S)$ is the set of all linear combinations built using a **finite** number of vectors from $S$. To get a generic element of $S$, choose some positive integer $k$, choose $k$ vectors from $S$, and then form the linear combination: $\mathbf{x} = \sum_{j=1}^k c_j\mathbf{v}_j$ where $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k \in S$ and where $c_1, c_2, \ldots, c_k \in \mathbb{R}$.

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  - Give a generic element of $span(\mathbf{u}, \mathbf{v}, \mathbf{w})$.
  (Hint: It is not $c\mathbf{u} + c\mathbf{v} + c\mathbf{w}$, why not?)
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- Jeff Stuart     Pacific Lutheran University     jeffrey.stuart@plu.edu