The Fundamental Theorem of Linear Algebra

Gilbert Strang
MIT

Big Picture: Column space and nullspace of A and A^T

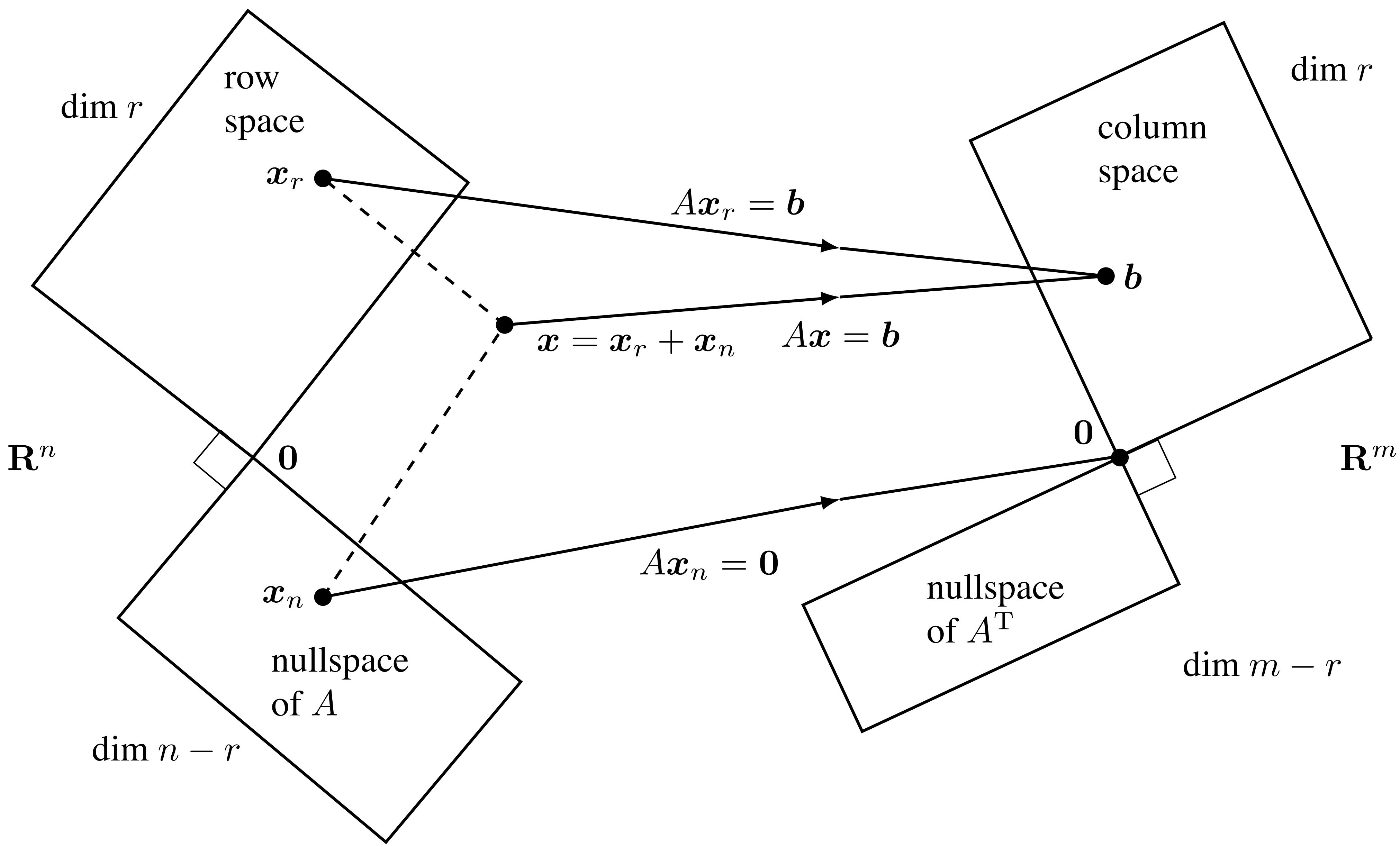

Figure 1: The action of A: Row space to column space, nullspace to zero.

$$m > n \text{ in } Ax = b \qquad \text{Solve } \; A^T A\hat{x} = A^T b$$

$$\text{Projection} \qquad p = A\hat{x} = A(A^T A)^{-1} A^T b = Pb$$

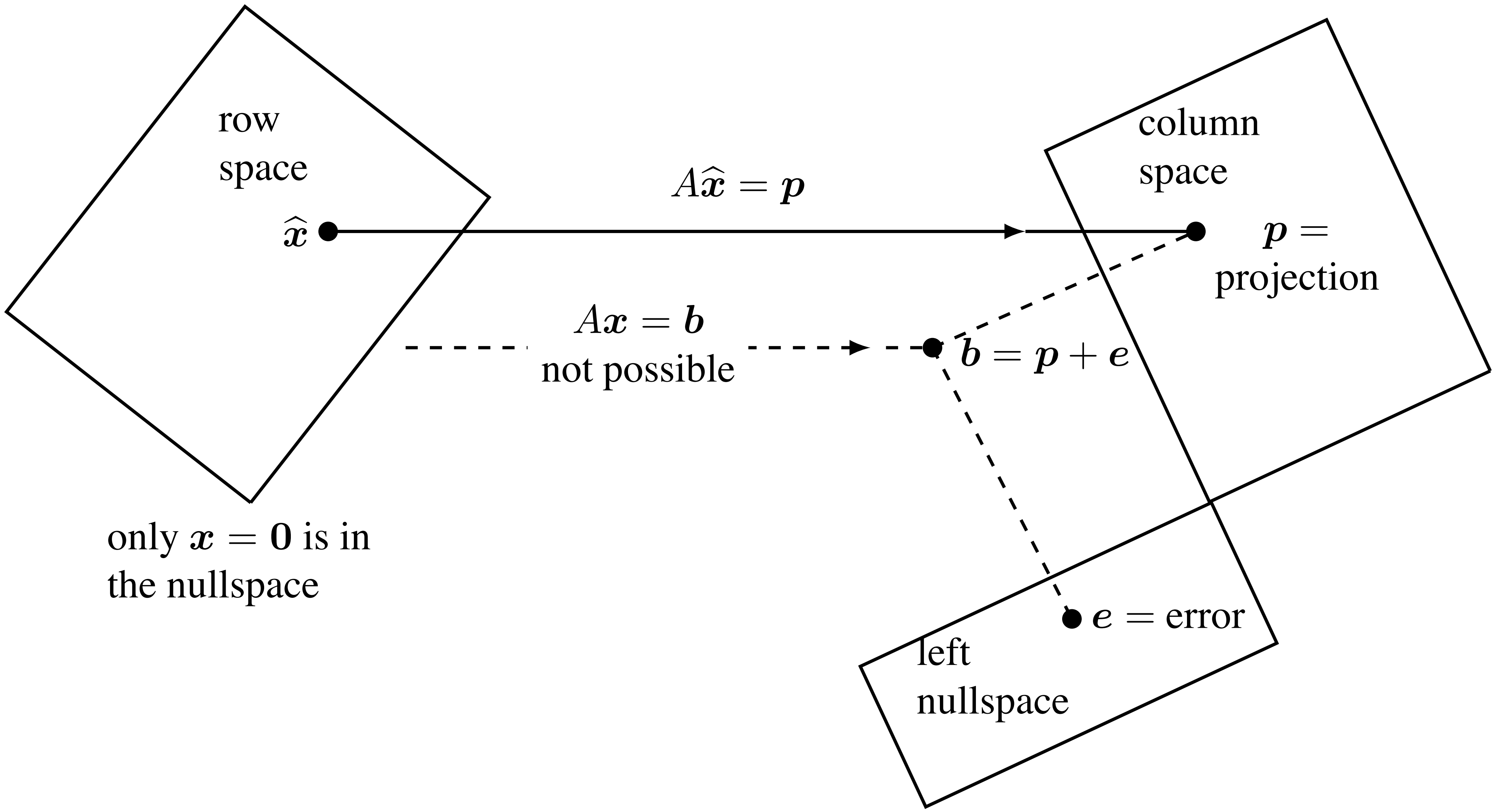

Figure 2: Least squares: $\widehat{\boldsymbol{x}}$ minimizes $\|\boldsymbol{b} - A\boldsymbol{x}\|^2$ by solving $A^{\mathrm{T}} A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$.

SVD = Singular Value Decomposition

$A = U \; \Sigma \; V^T$ (m x m) (m x n) (n x n)

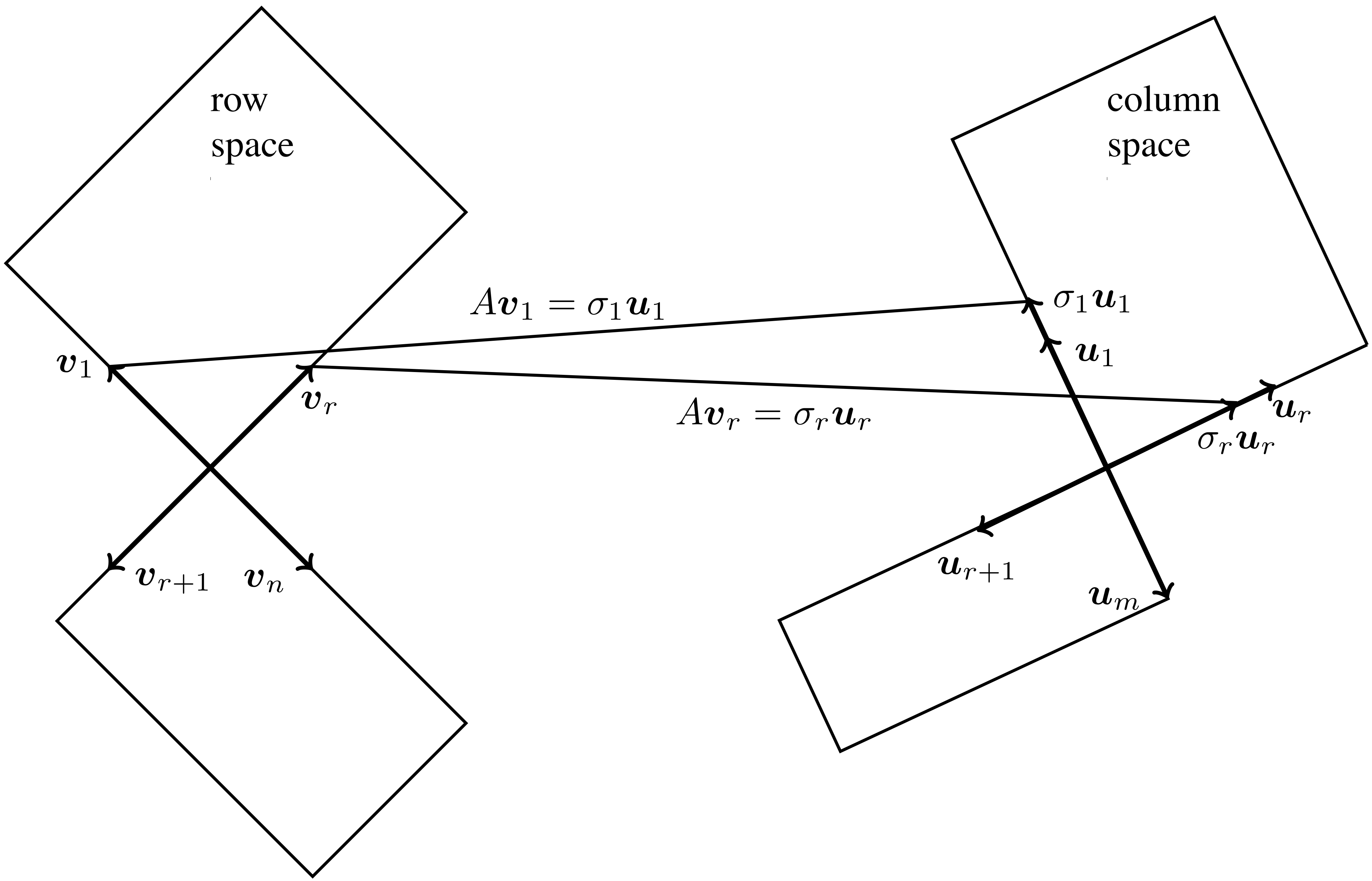

Figure 3: Orthonormal bases that diagonalize A.

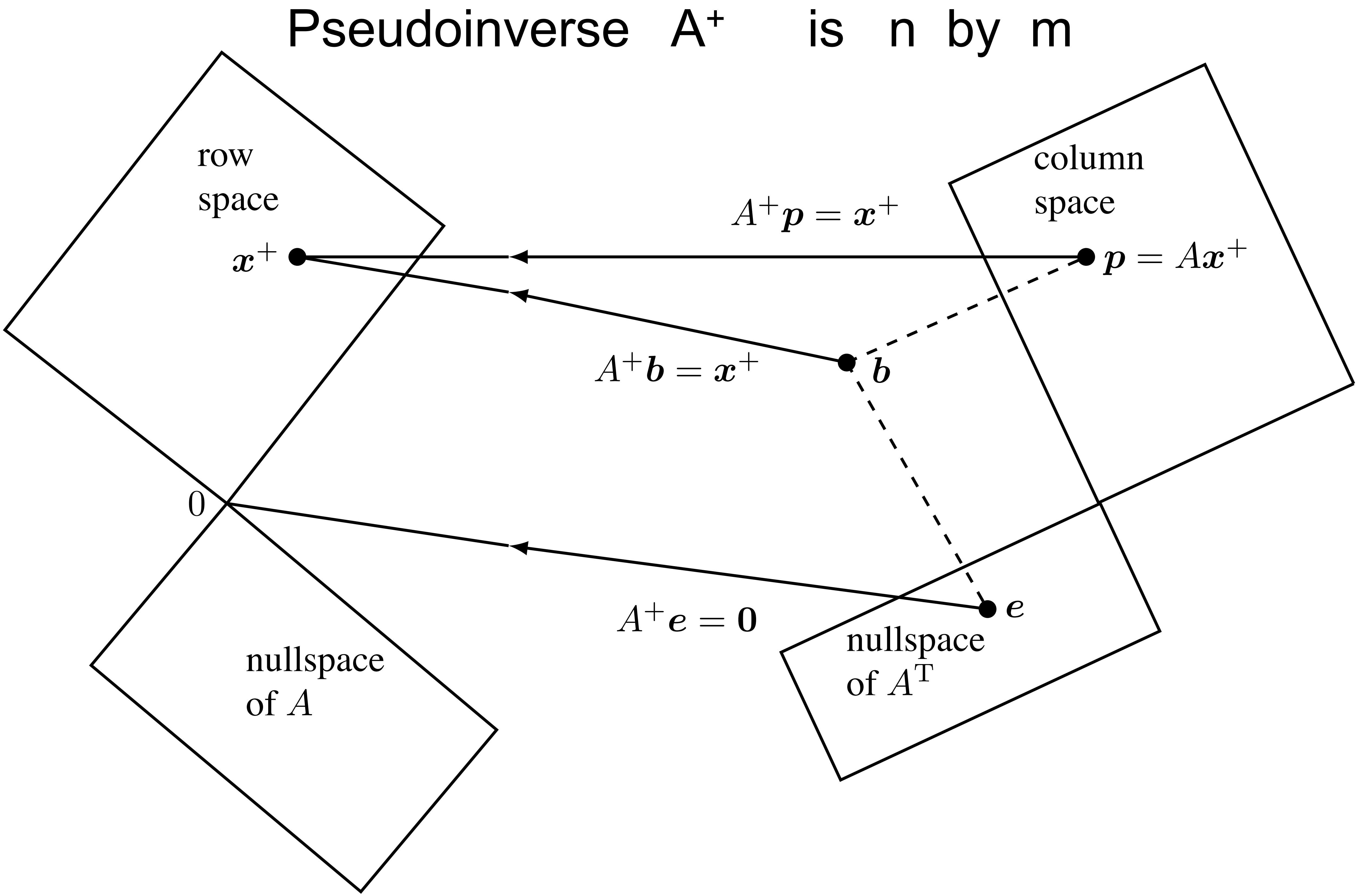

Figure 4: The inverse of A (where possible) is the pseudoinverse A^+.

SVD

Construct V, Σ, U in $A = U \Sigma V^T$

$v_1, \ldots, v_r$ orthonormal eigenvectors of $A^T A$

$$A^T A v_i = \lambda_i v_i \qquad \lambda_i = \sigma_i^2 \qquad \lambda_i > 0$$

KEY $u_i = A v_i$ are orthogonal because

$$(A v_i)^T (A v_j) = v_i^T (A^T A v_j) = \lambda_j v_i^T v_j$$

Normalize to length 1 Divide u_i by $\sigma_i = ||u_i||$

Choose $v_{r+1}, \ldots, v_n$ orthonormal in $N(A)$

Choose $u_{r+1}, \ldots, u_n$ orthonormal in $N(A^T)$

Then $A v_1 = \sigma_1 u_1 \quad \ldots \quad A v_r = \sigma_r u_r$

An all-black image (300 × 300 pixels):

$$\underbrace{\overset{300\times 300}{\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}}}_{\substack{\text{Do not send this} \\ (300)^2 \text{ data}}} = \underbrace{\overset{300\times 1}{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}} \overset{1\times 300}{\begin{bmatrix} 1 \cdots 1 \end{bmatrix}}}_{\text{Send this instead!}}$$

Grey-scale image (300 × 300 pixels):

$$\frac{1}{300}\begin{bmatrix} 0 & 1 & \cdots & 299 \\ \vdots & \vdots & & \vdots \\ 0 & 1 & \cdots & 299 \end{bmatrix} = \frac{1}{300}\underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & \cdots & 299 \end{bmatrix}}_{\text{Send this!}}$$

French flag (300 × 300 pixels):

$$\begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 0 \cdots 0 & 1 \cdots 1 & 0 \cdots 0 \end{bmatrix}$$

row rank = column rank

Factor $A_{m \times n} = C_{m \times r} D_{r \times n} = [c_1 \ldots c_r][d_1 \ldots d_n]$

Basis for column space in C: dim r

Coefficients for each column are in D

Look again, REVERSED $A = \begin{bmatrix} \text{row } 1 \\ \vdots \\ \text{row } m \end{bmatrix} \begin{bmatrix} \text{row } 1 \\ \vdots \\ \text{row } r \end{bmatrix}$

A = CD expresses rows of A by rows of D

Coefficients for each row are in C

Then row space has dimension $\leq r$

row rank = column rank

Start $x_1, \ldots, x_r$ basis for row space

Show $Ax_1, \ldots, Ax_r$ independent in column space

Suppose $0 = c_1 A x_1 + \cdots + c_r A x_r$

$$= A(c_1 x_1 + \cdots + c_r x_r) = Av$$

v is in row space and null space: $v = 0$.

Then $c_i = 0$ since x_i are a basis.