Singular Value Decomposition: A Thrilling Inspiration in Linear Algebra

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Definition

- The SVD of an m by n matrix A is written as

$$A = U\Sigma V^T$$

where U is the $m \times m$ orthogonal matrix, V is the $n \times n$ orthogonal matrix, and Σ is the $m \times n$ orthogonal matrix having the singular values $\sigma_1 \geq \sigma_2 \geq . . . \geq \sigma_{\min(m,n)}$ of A in order along its diagonal. The columns of U and V are called the left and right *singular vectors* for A.

The relative size of the matrices U, Σ, and V when m>n

When m>n

$$
\underbrace{\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}}_{\text{A}}
=
\underbrace{\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}}_{\text{U}}
\underbrace{\begin{bmatrix} \bullet & & \\ & \bullet & \\ & & \bullet \\ & & \\ & & \end{bmatrix}}_{\Sigma}
\underbrace{\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}}_{V^T}
$$

The relative size of the matrices U, Σ, and V when m<n

When m<n

$$\underbrace{\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \bullet & & & & \\ & \bullet & & & \\ & & \bullet & & \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}}_{V^T}$$

A *be an* $m \times n$ *matrix*

$A \longrightarrow \mathbb{R}^n$ *into* $\mathbb{R}^m$

$$AV = U\Sigma$$

- The columns of V and U provide the basis for the domain and range.

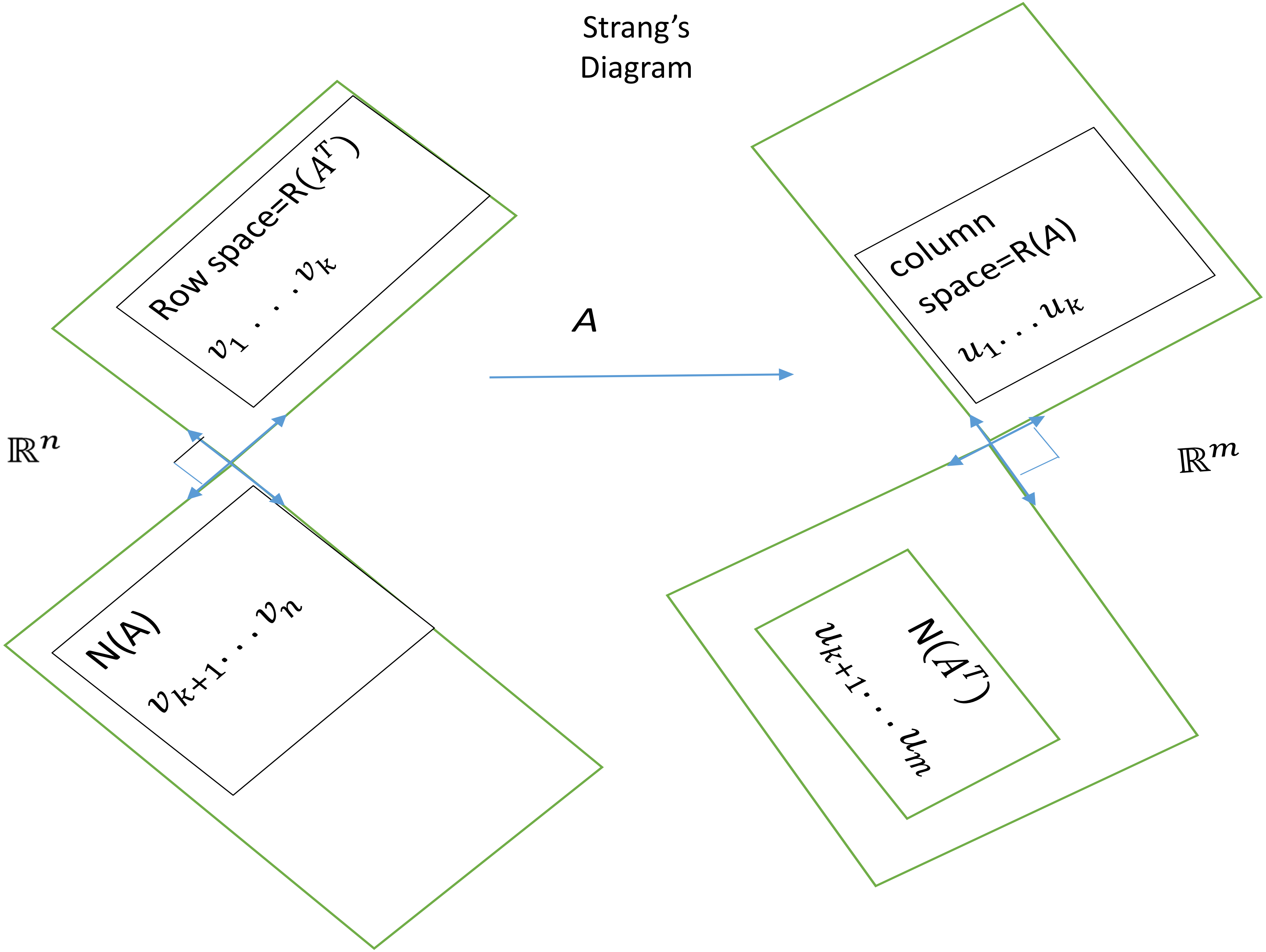
- U is an m by m orthogonal matrix. Its columns $u_1, ...u_k . . . .u_m$ are basis vectors for the column space and the left null space.
- V is an n by n orthogonal matrix. Its columns $v_1, ...v_k . . . .v_n$ are basis vectors for the row space and the null space.

$$Ax = 0$$

$$Ax = \begin{bmatrix} -\mathrm{row}\ \ 1- \\ -\mathrm{row}\ \ 2- \\ -\mathrm{row}\ \ \mathrm{m}- \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The nullspace *N(A)* is orthogonal to the row space $R(A^T)$
The left nullspace $N(A^T)$ is orthogonal to the column space $R(A)$

Strang's Diagram



Observations

Let A be an $\mathrm{m} \times n$ matrix with a singular value decomposition $U\Sigma V^T$

- The left and right singular vectors u_i and v_i tend to become more and more oscillatory as index *i* increases, i.e. as σ_i decreases.
- The matrix V is obtained from the diagonal factorization, $A^T A = V\Sigma V^T$. V diagonalizes $A^T A$, it follows that the v_j's are the eigenvectors of $A^T A$. Similarly U diagonalizes AA^T.

- $A^T A = V\Sigma^T U^T U\Sigma V^T = V \begin{bmatrix} \sigma_1^2 & & & & \\ & \ddots & & & \\ & & \sigma_r^2 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix} V^T$

- Hence $\sigma_1^2, \ldots, \sigma_r^2$ (and 0 if r < n) are eigenvalues of $A^T A$ and the columns of V are the the eigenvectors of $A^T A$.
- The singular values are the square roots of the nonzero eigenvalues of both $A^T A$ and AA^T.

- If A has rank *r*<*n*, if we set

$$U_1 = (u_1, \ldots, u_r) \qquad V_1 = (v_1, \ldots, v_r)$$

and define

$$\Sigma_1 = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_r \end{bmatrix},$$

then $\boldsymbol{A} = \boldsymbol{U_1} \Sigma_{\boldsymbol{1}} \boldsymbol{V_1^T}$ is called the compact form of the singular value decomposition of A.

Applications_1
Numerical Rank

- It is necessary to determine the rank of a matrix.
- If the computations involve a nonsingular matrix that is very close to being singular, then the computed solutions may have no digits of accuracy.
- The singular values provide a way of measuring how close a matrix to matrices of lower rank.

Numerical Rank

- The numerical rank of an m x n matrix is the number of singular values of the matrix that are greater than $\sigma_1 \max(m, n)\, \epsilon$, where σ_1 is the largest singular value of A and ϵ is the machine epsilon.

Example

Suppose that a 5x5 matrix with singular values σ_1=4; σ_2=1; σ_3=10^{-12};

σ_4=3.1x10^{-14}; σ_1=2.6x10^{-15}; and suppose that the machine epsilon

(ϵ) is 5x10^{-15}.

Solution: $\sigma_1 \max(m, n)\,\epsilon$=4x5x5x10^{-15} =10^{-13}

Since three of the singular values are greater than 10^{-13}

The matrix has numerical rank 3

Applications_2
Image Compression

Digital image may be considered as matrices whose elements are the pixel values of the image.

- Computational time
- Reduces the cost in image storage and transmission

An m by n matrix *A* *An* matrix of lower rank r *where*
$r<min(m,n)$

Image Compression

- Amount of storage required for a regular image= mn

- If the image A has singular value decomposition $U\Sigma V^T$, then A can be represented by the outer product expansion

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + ... + \sigma_n u_n v_n^T$$

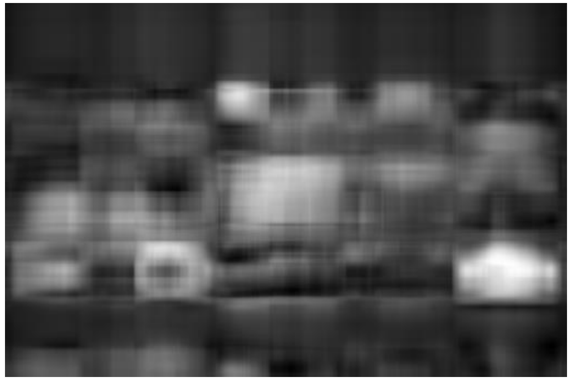
The closest matrix of rank ***r*** is obtained by truncating the sum after first r terms

$$A_r = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + ... + \sigma_r u_r v_r^T$$

Amount of storage required = $r(m + n + 1)$

Original 384 by 512 image

Rank 5 Approx. to image



Rank 10 Approx. to image

Rank 43 Approx. to image

Applications_3
Deblur an image

- The relation between a blurred and a sharp image is given by

$$A_c X A_r^T = B$$

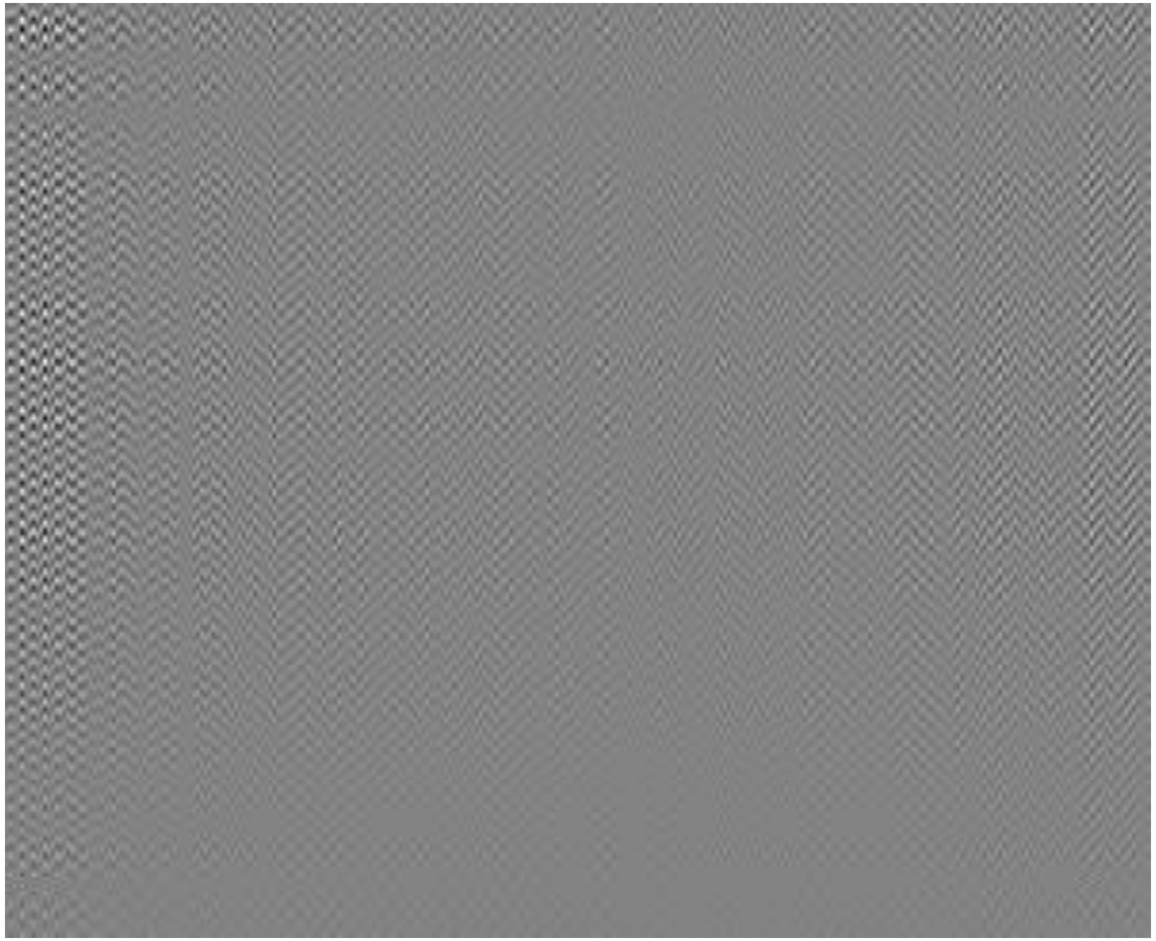
$A_c \in \mathbb{R}^{mxm}$ $\Rightarrow$ vertical blurring

$A_r \in \mathbb{R}^{nxn}$ $\Rightarrow$ horizontal blurring

Blurred image

$$X_{naive} = A_c^{-1} B A_r^{-T}$$

Xnaive



$$B = B_{exact} + \mathrm{E}$$

1. Blurred image is collected by a mechanical device.
2. When the image is converted from analog to digital, it is represented by a finite number of digits.

$$A_c X A_r^T = B_{exact} + \mathrm{E}$$

$$X_{naive} = A_c^{-1} B A_r^{-T} + A_c^{-1} E A_r^{-T}$$

Inverted noise

Pseudoinverse of A

The pseudoinverse of A=$U\Sigma V^T$ is

$$A^{\dagger}=V\Sigma^{\dagger}U^T$$

- The pseudoinverse $A^{\dagger}$ agrees with A^{-1} when A is invertible. The solution is

$$X_k = (A_c)_k^{\dagger} B\left((A_r)_k^{\dagger}\right)^T + \text{Inverted noise}$$

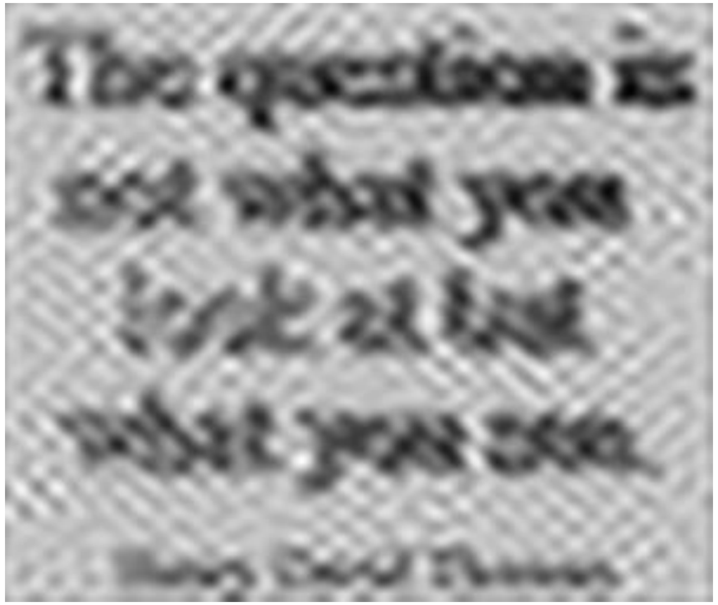
The diagonal entries are of the form $\left(\Sigma^{\dagger}\right)_{ii}$=$\begin{cases} \frac{1}{\sigma_i} & if \quad 1 \le i \le k; \\ 0 & if \quad k+1 \le i \le n \end{cases}$

Singular vectors that corresponds to smaller singular values represent high frequency information. Since the high frequency components are dominated by error, we ignore the high frequency components and we have the following truncated expansion

$$A_k^{\dagger} = \sum_{i=1}^{k} \frac{1}{\sigma_i} v_i u_i^T$$

Blurred image

Reconstructed Image



Using k=45, we got the reconstructed image. The Blurred image, Ar, Ac are given in challenge2.mat. This can be found at www.siam.org/books/fa03.

MATLAB code

- function out = pseudosum(Ac,Ar,B,K)
- % K is the no. of singular values to be chosen.
- [u1, s1, v1] = svd(Ac);
- [u2, s2, v2] = svd(Ar);
- s1i = inv(s1);
- s2i = inv(s2);
- Ac1=v1(:,1:K)*s1i(1:K,1:K)*u1(:,1:K)';
- Ar1=v2(:,1:K)*s2i(1:K,1:K)*u2(:,1:K)';
- out = Ac1*B*Ar1'
- figure, imshow(out, []);
- end

References

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Thank you