Motivating Students for Linear Algebra by using Puzzles

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Introduction

- ❖ Puzzle 1: ***ADDITION TABLE***
- Purpose: To learn rank, free variables, and consistency
- ❖ Puzzle 2: ***GOSSIP NETWORK***
- Purpose: To understand the application of matrix multiplication

Given a, b, c, and d, find all possible values of r,s,t and u to fill in the following addition table

+	t	u
r	a	c
s	b	d

(1)

We have

$$\begin{cases} r + t = a \\ s + t = b \\ r + u = c \\ s + u = d \end{cases}$$

(2)

Writing equations (2) for the puzzle in (1) in matrix form gives

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} r \\ s \\ t \\ u \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

The echelon form of matrix A is

Rank=# of nonzero rwows=3

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \\ u \end{bmatrix} = \begin{bmatrix} a \\ b \\ a-c \\ d-b-c+a \end{bmatrix}$$

Number of free variables=number of unknowns - rank(A)=4-3=1

The system becomes consistent

$$d-b-c+a=0 \quad \Longleftrightarrow \quad a+d=b+c$$

Infinite number of solutions

Example : 2X2 Puzzle

+	t	u
r	3	5
s	4	6

→

+	-2+u	u
5-u	3	5
6-u	4	6

↓ Let u=1

+	-1	1
4	3	5
5	4	6

Example: 3X3 Puzzle

+	u	v	w
r	6	3	5
s	5	2	4
t	7	4	6

→

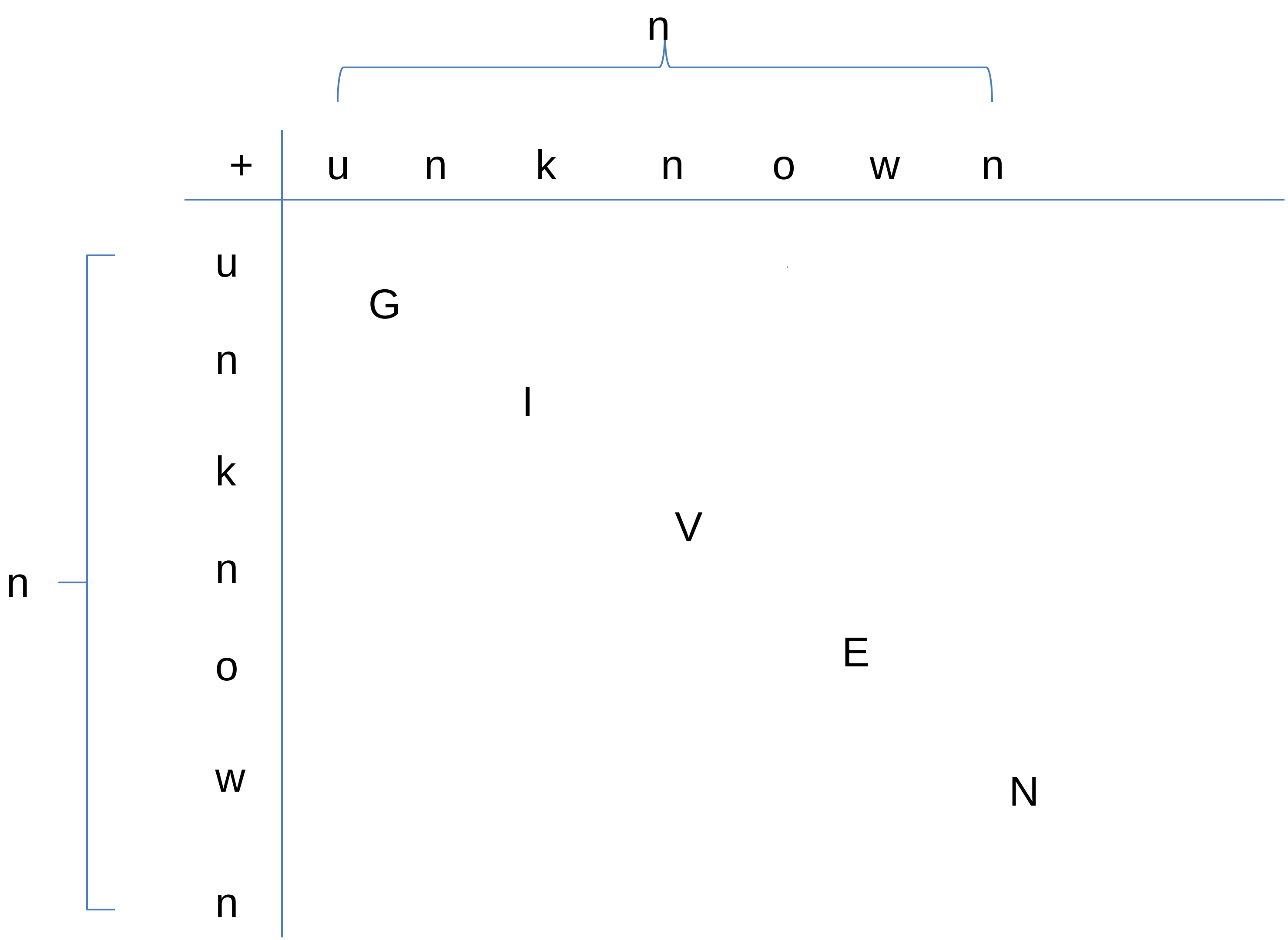
+	1+w	-2+w	w
5-w	6	3	5
4-w	5	2	4
6-w	7	4	6

→ Let w=1

3 X 3 Puzzle

+	2	-1	1
4	6	3	5
3	5	2	4
5	7	4	6

Generalization of the $n \times n$ puzzle



$$n + n = 2n$$ unknowns

$$n \times n = n^2$$ equations

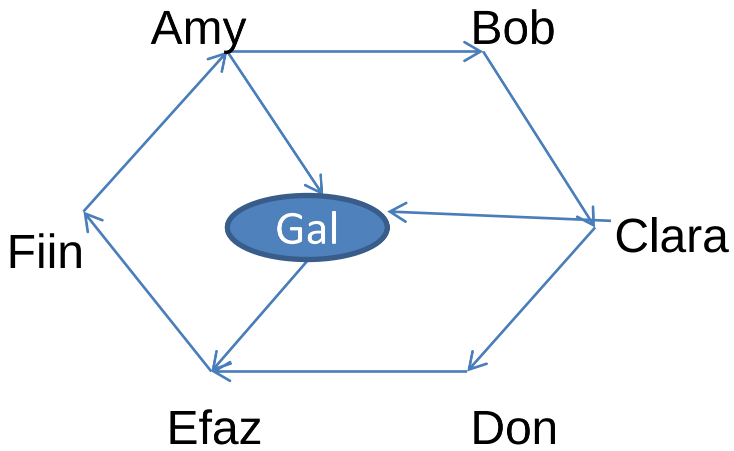
$$(n-1) \times (n-1) = (n-1)^2$$ No. of 2 x 2 blocks

Rank (A)= $n^2 - (n-1)^2 = 2n-1$

No. of free variables= $2n-(2n-1)=1$

Puzzle 2: Seven people are all connected by e-mail. Whenever one of them hears a juicy piece of gossip, he or she passes it along by emailing it to someone else in the group according to following table:

Sender	Recipients
Amy	Bob, Gal
Bob	Clara
Clara	Don, Gal
Don	Efaz
Efaz	Fiin
Fiin	Amy
Gal	Ehaz



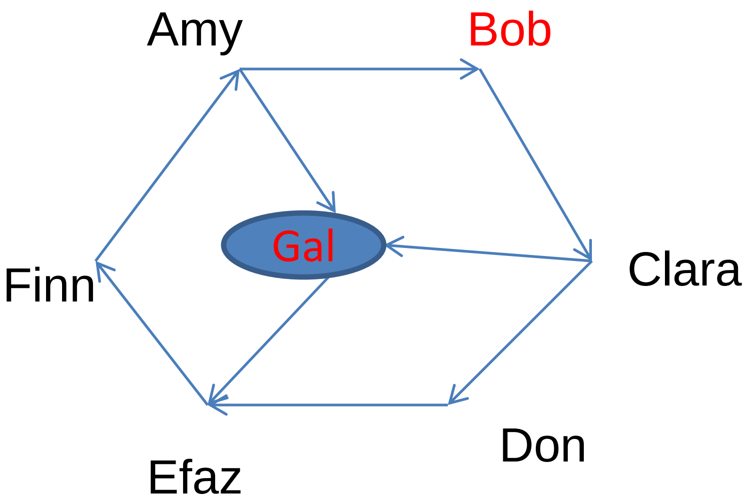
Diagraph G

If Amy hears a rumor, how many paths will it take for everyone else to hear the rumor?

Adjacency matrix: If **G** is a diagraph with n vertices, then the adjacency matrix is defined by

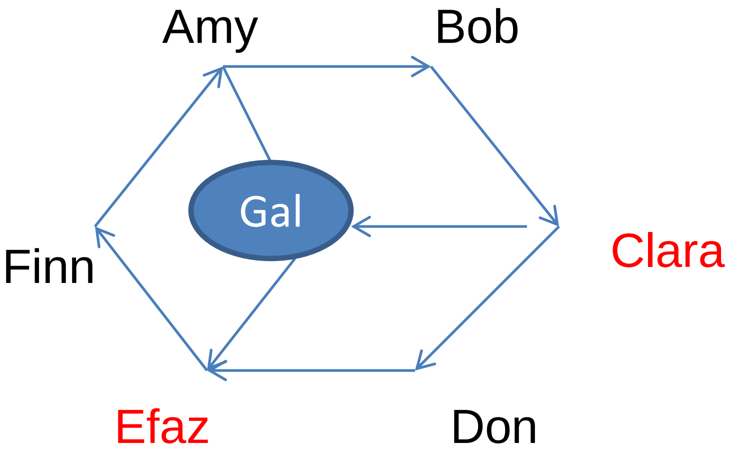
$$a_{ij} = \begin{cases} 1 & \text{if there is a directed edge between vertices } \boldsymbol{i} \text{ and } \boldsymbol{j} \\ 0 & \text{otherwise} \end{cases}$$

Letting people **Amy** through **Gal** correspond to vertices 1 through 7 in alphabetical order, the adjacency matrix is



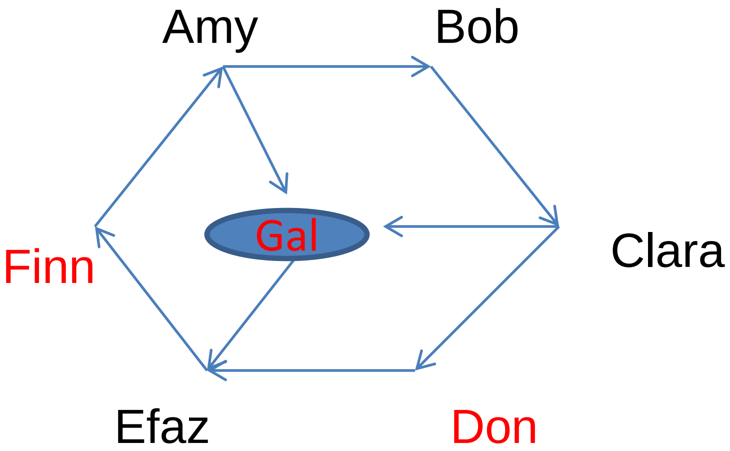
Bob Gal

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$



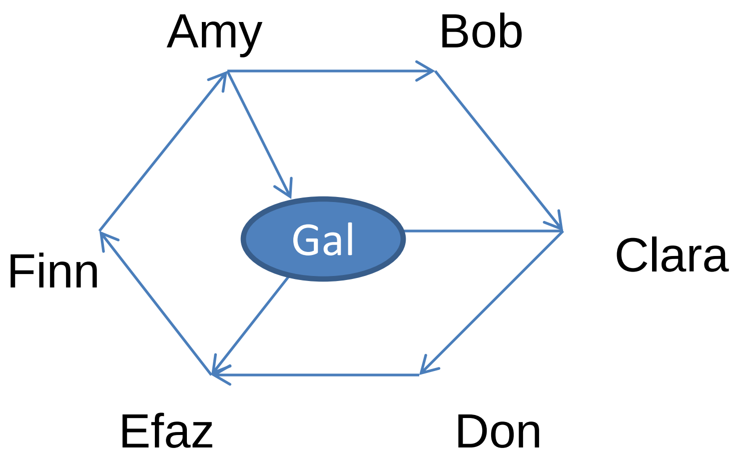
Clara ★ Efaz ★

$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



Don Finn Gal

$$A^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



$$A + A^2 + A^3 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

This puzzle is reminiscent of the notion of “six degrees of separation”, which suggests that any two people in the world are connected by a path of acquaintances, whose average length is six.

Watts-Strogatz model

Average path length between two nodes in a random network

$$= \ln N / \ln K$$

N=total nodes

K= acquaintances per node

$N = 300{,}000{,}000$

90% of the US population

$K = 30$

Acquaintances per person

Average path length= $\,^{19.5}\!/\!_{3.4}$ =5.7

$N = 6{,}000{,}000{,}000$

90% of the World population

$K = 30$

Acquaintances per person

Average path length= $^{22.5}\!/_{3.4}$ = 6.6

Bibliography

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- Duncan J. Watts & Steven H. Strogatz, *Collective dynamics of ‘small-world’ networks*, Nature, **393** (1998), 440-442.

Thank You