# Magic Squares and other Explorations in Linear Algebra

Michelle Ghrist

United States Air Force Academy

# Introduction

- Math majors class
  - Students: 19 (sophomores-seniors) with a wide variety of backgrounds and abilities
  - Proof-based (but with applications)
  - Textbook: Stephen Lay's Linear Algebra and its applications ($4^{th}$ ed)
  - Students have access to and experience with MATLAB and Mathematica
  - My first time teaching this course

# “Extra” Goals

- Develop better proof-writing skills
- Encourage students to see the bigger picture of linear algebra
  - How do various ideas connect?
  - Ask deep questions (research-like mentality)
- Appreciate some of the many applications of linear algebra
- Lots of hands-on learning: groupwork, “labs”

# Idea #1: Set the tone

- Lesson 1 brainstorming in groups:
  - In what contexts have you heard the word "linear"?
  - In what applications/other classes have you used matrices?
- Discussion of how this course connects to other courses
- Start to establish connections/big picture

# Idea #2: commuting matrices (Lesson 6)

- Context: discussion of elementary matrices (plus rotation and projection matrices)
  - Left-multiplying a matrix by an elementary matrix performs a single elementary row operation.

Example: $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 9 & 12 & 15 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

  - Question posed to the students: what happens if I right-multiply a matrix by an elementary matrix?

Example: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 3 \\ 4 & 13 & 6 \\ 7 & 22 & 9 \end{pmatrix}$

# Idea #2: commuting matrices

- Elementary (plus rotation and projection) matrices
  - Question posed by a student: if I left-multiply and right-multiply a matrix by one of these five matrices, do I ever get the same result?

# Idea #3: Lesson 7 quiz

1. Let $T$ be a linear transformation from $\mathbb{R}^n$ onto $\mathbb{R}^m$. What else can you conclude about $T$ and its associated standard matrix $A$? (List as many ideas as you can.) Denote which of these are iff's.

- Each $\vec{b} \in \mathbb{R}^m$ is the image of at least one $\vec{x} \in \mathbb{R}^n$ under $T$. (iff)
- $A$ has $m$ pivot columns. (iff)
- $A$ is $m \times n$ with $m \leq n$.

2. "Let $T$ be a one-to-one linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ …"

# Idea #4: LU Decompositions (Lesson 9)

- Write $A = LU$ where
  - $L = m \times m$ lower triangular matrix with 1's on the diagonal
  - $U = m \times n$ echelon form of $A$

# Idea #5: Matrix Factorizations (Lesson 10)

- Discuss $LU$ (in-depth), $QR*$, and SVD* (briefly)

- Pre-class question posed to students:

  "Via research, find at least one other matrix factorization."

* Explored further in later lessons

# Idea #6: Magic Squares: Lesson 18

- Inspiration: problem from Strang's Linear Algebra and its Applications
- Let $V = M_{2\times 2}$, and let $W$ be the subspace of $V$ whose row sums and column sums are all equal.
  - Prove that $W$ is a subspace of $V$.
  - Find a basis for $W$.
  - Challenge: repeat if $V = M_{3\times 3}$.

# Idea #7: Project (last year)

- Semester long individual research project
  - Choose from list of 28 projects (must be new to you)
  - Find at least 3 references; write paper and give presentation.

# Idea #8: Project (this semester)

- Individual research project, focused on second part of semester
- Locate a journal article in which one of the ideas from class is applied (non-math journal)
  - E.g., matrix factorization, change of coordinate matrix
- Test/recreate results (use technology!)
- Write a paper summarizing:
  - Application topic (may require outside research)
  - How the idea was used and your results.

# Conclusions

- Informally, students developed a better appreciation for the inter-connectedness of linear algebra.
- Many students enjoyed the lab exercises, seeing some of the many applications of linear algebra, and being allowed to think deep thoughts.

- Thank you for attending.
- Any questions?

Michelle.Ghrist@usafa.edu