Spanning Sets and Linear Independence
Discovering Shortcuts and Connections

Warm-up Problems: Be sure to show your work and write down any justifications or observations carefully on a separate sheet as you will be referring to this examples in drawing conclusions later in the worksheet.

Consider the sets of vectors $S = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} \right\}$, $T = \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} \right\}$, and $U = \left\{ \begin{bmatrix} 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \end{bmatrix} \right\}$.

1. Does the vector $\mathbf{x} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$ lie in Span(*S*)? in Span(*T*)? Does $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ lie in Span(*U*). If it does, write **x** or **y** as a linear combination of the vectors in the given set.

2. Determine whether $\text{Span}(S) = \mathbb{R}^3$, whether $\text{Span}(T) = \mathbb{R}^3$, and whether $\text{Span}(U) = \mathbb{R}^2$.

3. Which of the sets *S*, *T*, and *U* are linearly independent?

Possible shortcuts: Decide whether each of the following approaches is valid and give a reason for your answer. When you are convinced of shortcuts (or want to warn yourself about non-shortcuts), you might want to enter these into your course notes.

1. In determining whether **x** lies in Span(*S*), Jeremy sets up and solves the augmented matrix $\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & -3 & -1 \\ 2 & 0 & 3 & 3 \end{array}\right]$.

2. In determining whether **x** lies in Span(*T*), Elaine sets up and solves the augmented matrix $\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 4 & 1 & -1 \\ 0 & -1 & 3 \end{array}\right]$.

3. In determining whether **x** lies in Span(*T*), Jack sets up and solves the augmented matrix $\left[\begin{array}{cc|c} 1 & 4 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & 3 \end{array}\right]$.

4. In determining whether $\text{Span}(S) = \mathbb{R}^3$, Jill row-reduces the matrix $\begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ without column $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

   What must she notice about the row reduced form of the matrix, in order to determine whether $\text{Span}(S) = \mathbb{R}^3$?

5. Steve sets up the augmented matrix $\left[\begin{array}{ccc|c} 2 & 0 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$ to determine whether the vectors in S are linearly independent, but Mark argues that the last column of zeros is unnecessary. Which is right? Why?

6. Jayden sets up the augmented matrix $\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 1 & 8 & 4 & 0 \end{array}\right]$ to determine whether the vectors in U are linearly independent. Will this affect her result? Why or why not?

7. Tomas sets up the augmented matrix $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 2 & 0 & 3 & 0 \end{array}\right]$ to determine whether the vectors in S are linearly independent. Will this affect his result? Why or why not?

8. Of the nine questions (3 per number) in the section "Warm Up Problems", which two could have been answered without performing any computations (not even mental computations)? Why?

<u>Making Connections</u>: You will be given a stack of index cards that identify certain properties. Determine under which of the following headings each card belongs. Note that each card will fit under one of #1-3 and one of #4-6.

Throughout this activity, let $A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{bmatrix}$ where each $\mathbf{v}_i$ is a vector in $\mathbb{R}^n$.

1. The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k$ span $\mathbb{R}^n$.
2. The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k$ do not span $\mathbb{R}^n$.
3. We need more information to determine whether the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k$ span $\mathbb{R}^n$.
4. The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k$ are linearly dependent.
5. The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k$ are linearly independent.
6. We need more information to determine whether the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k$ are linearly dependent.