Putting Together the Puzzle: Understanding Linear Independence, Spanning, and Bases via Group Exploration

MAA Session on Innovative and Effective Ways to Teach Linear Algebra

Joint Mathematics Meetings, Baltimore MD

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Linear Algebra Course

1. Vectors
2. Systems of Linear Equations
    2.3 Spanning Sets and Linear Independence
3. Matrices
4. Eigenvalues and Eigenvectors

Poole, David *Linear Algebra: A Modern Introduction*

Linear Independence and Spanning Sets

- Challenging for students
- Often occur together in the course timeline
- Students tend to blend them together and need extra guidance in recognizing the distinct characteristics associated with each.

Exploring the Topic

- Introductory presentation with definitions and examples
- Group Work in three parts:
  - Warm up
  - Looking for Short Cuts
  - Making Connections

Warm Up—Routine Problems

Students are given three sets of vectors and asked whether

- each set spans $\mathbb{R}^n$,
- is linearly independent, and
- whether a given vector $\mathbf{b}$ lies in the span of one of the sets.

They are expected to use Gaussian elimination to solve $A\mathbf{u} = \mathbf{x}$, $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$ where A is the column matrix of vectors.

Looking for Shortcuts

Referring back to the sets of vectors in the warm-up, students are asked whether they can swap columns or rows in the matrix $\boldsymbol{A}$ and whether it is necessary to include a column of zeros or a column of variables for $\mathbf{b}$.

Sample Problems

Decide whether each of the following approaches is valid and give a reason for your answer. When you are convinced of shortcuts (or want to warn yourself about non-shortcuts), you might want to enter these into your course notes.

Consider $S = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} \right\}$ and $\mathbf{x} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$.

1. In determining whether **x** lies in Span(*S*), Jeremy sets up and solves the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & -3 & -1 \\ 2 & 0 & 3 & 3 \end{array}\right]$$

Decide whether each of the following approaches is valid and give a reason for your answer. When you are convinced of shortcuts (or want to warn yourself about non-shortcuts), you might want to enter these into your course notes.

Consider $T = \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} \right\}$ and $\mathbf{x} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$.

2-3. In determining whether **x** lies in Span(*T*), Elaine and Jack set up and solve the augmented matrices

$\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 4 & 1 & -1 \\ 0 & -1 & 3 \end{array}\right]$ and $\left[\begin{array}{cc|c} 1 & 4 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & 3 \end{array}\right]$ respectively.

4. Jill row-reduces the matrix $\begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ without the column $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. What must she notice about the row reduced form of the matrix, in order to determine whether $\text{Span}(S) = \mathbb{R}^3$?

5. Steve sets up the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 0 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

to determine whether the vectors in

$$S = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} \right\}$$

are linearly independent, but Mark argues that the last column of zeroes is unnecessary. Which is right? Why?

6. Jayden sets up the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 1 & 8 & 4 & 0 \end{array}\right]$$

to determine whether the vectors in

$$U = \left\{ \begin{bmatrix} 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \end{bmatrix} \right\}$$

are linearly independent. Will this affect her result? Why or why not?

7. Tomas sets up the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 2 & 0 & 3 & 0 \end{array}\right]$$

to determine whether the vectors in

$$S = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} \right\}$$

are linearly independent. Will this affect his result? Why or why not?

8. Of the nine questions (3 per number) in the section “Warm Up Problems”, which two could have been answered without performing any computations (not even mental computations)? Why?

You will be given a stack of index cards that identify certain properties. Determine under which of the following headings each card belongs. Note that each card will fit under one of #1-3 and one of #4-6.

Throughout this activity, let $$A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{bmatrix}$$

where each $\mathbf{v}_i$ is a vector in $\mathbb{R}^n$.

1. The vectors $\mathbf{v}_1, \mathbf{v}_2, ... \mathbf{v}_k$ span $\mathbb{R}^n$.

2. The vectors $\mathbf{v}_1, \mathbf{v}_2, ... \mathbf{v}_k$ do not span $\mathbb{R}^n$.

3. We need more information to determine whether the vectors $\mathbf{v}_1, \mathbf{v}_2, ... \mathbf{v}_k$ span $\mathbb{R}^n$.

Note that the boxes below are not necessarily in the correct column.

One of the vectors can be expressed as a linear combination of the others.

The row reduced form of the augmented matrix A has a leading entry in every column.

The linear system corresponding to $[\mathbf{A}|\mathbf{0}]$ has exactly one solution.

$\text{Rank}(A) = n$

$k = n$

The linear system corresponding to $[\mathbf{A}|\mathbf{0}]$ has at least two solutions.

$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \cdots + c_k\mathbf{v}_k = 0$ only if $c_i = 0$ for all i.

One vector is a scalar multiple of one of the others.

The row reduced form of the augmented matrix A has no all zero rows.

$k < n$

$n < k$

$\text{Rank}(A) < n$

4. The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k$ are linearly dependent.

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