Student Projects to Visualize Iteration Patterns of Matrices With Complex Eigenvalues

Rosemary Carroll Farley Ph.D.
rosemary.farley@manhattan.edu

MANHATTAN COLLEGE

Project One:

Motivation:

There was a diagram in our linear algebra book (Lay) that showed a graph of the iterates of the point x_0 under the action of the matrix A.

The proposed project:

Suppose A is a 2 x 2 matrix and x_0 is an arbitrary 2 x 1 vector in R^2 .

Let $x_1 = Ax_0$, $x_2 = Ax_1$, ..., $x_n = Ax_{n-1}$.

Then the iterates of the point x_0 under the action of the matrix A are

$$\{x_0, x_1, x_2, x_3, \ldots\} = \{x_0, Ax_0, A^2x_0, A^3x_0 \ldots\}$$

1. Provide a graph of the resulting iterates for any A and any x_0.

2. Predict the number of distinct iterates and prove your result.

The Four Step Method for Creating Animations:

Step One: Plot one component of the animation.

Step Two: Create and name the plots.

Step Three: Create a list of these plots.

Step Four: Display the animation.**

** OR show all of the created plots at the same time.

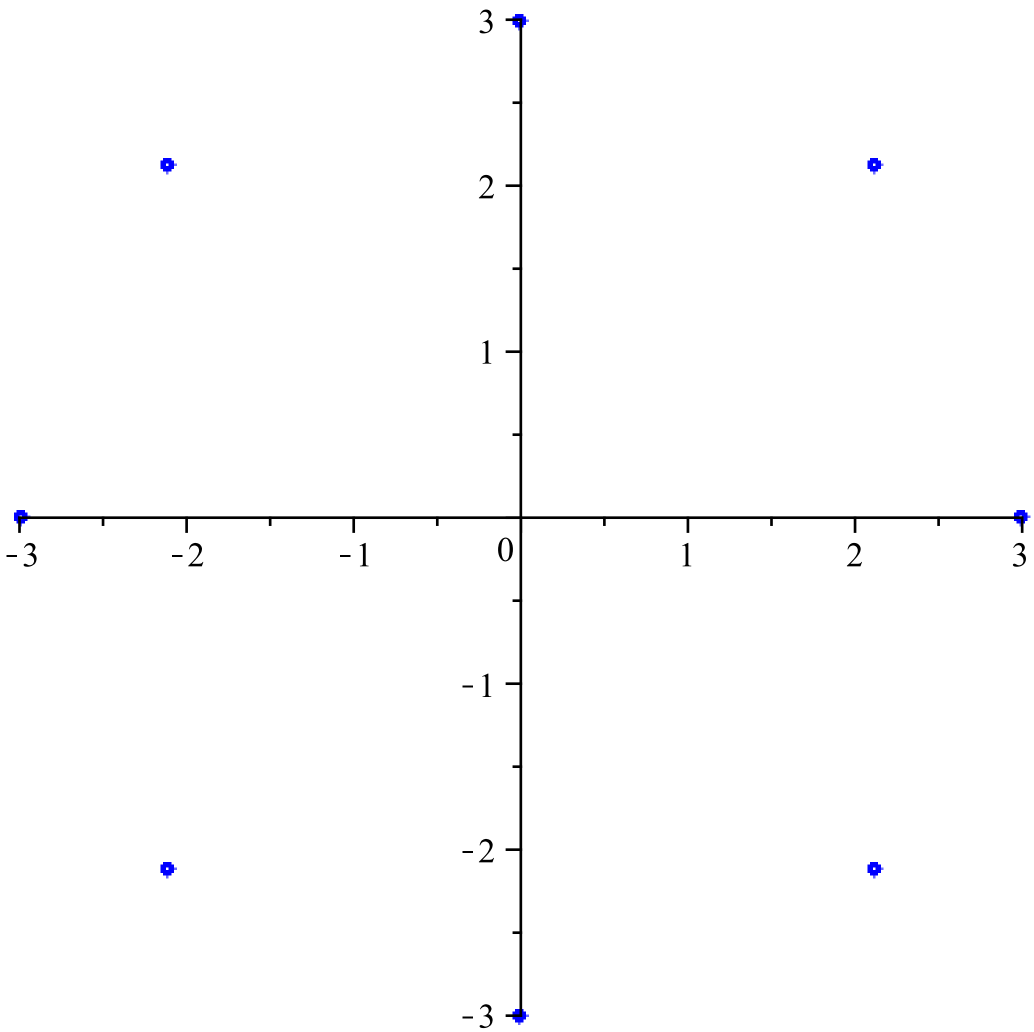
Farley, Rosemary & Tiffany, P. A Four Step Method for Creating Animations. Computers in Education Journal. Vol. 4 No. 3 July--September 2013, 59-62.

The Code:

```
for n from 1 to 100 do
    Q[n] := pointplot(x, symbolsize = 10, color = blue) :
    x := multiply(A, x) :
od:
L := [seq(Q[n], n = 1 ..100)] :
display(L)
```

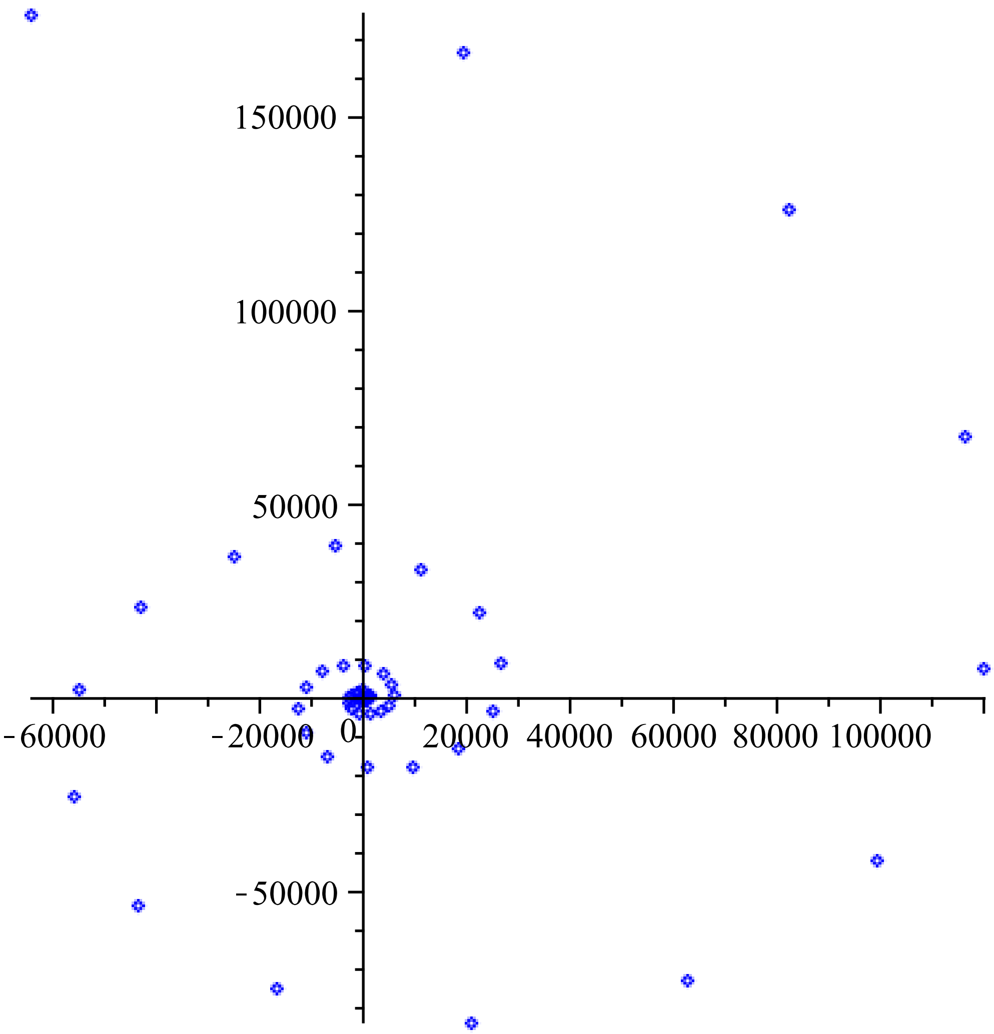
The Experimentation: Trial #1:

$$A := \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \quad \text{and} \quad x := \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$



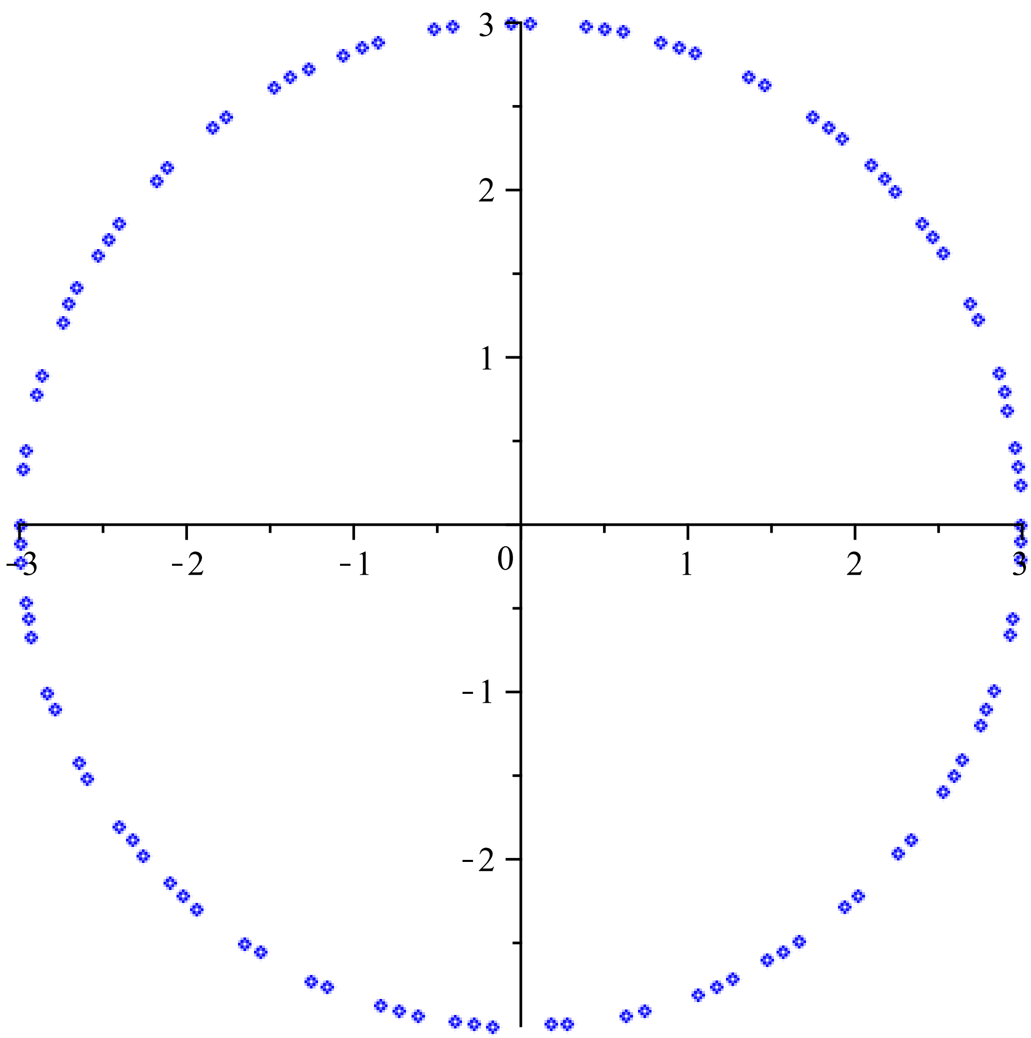
Trial #2:

$$A := \begin{bmatrix} 1 & -\frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \text{ and } x := \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$



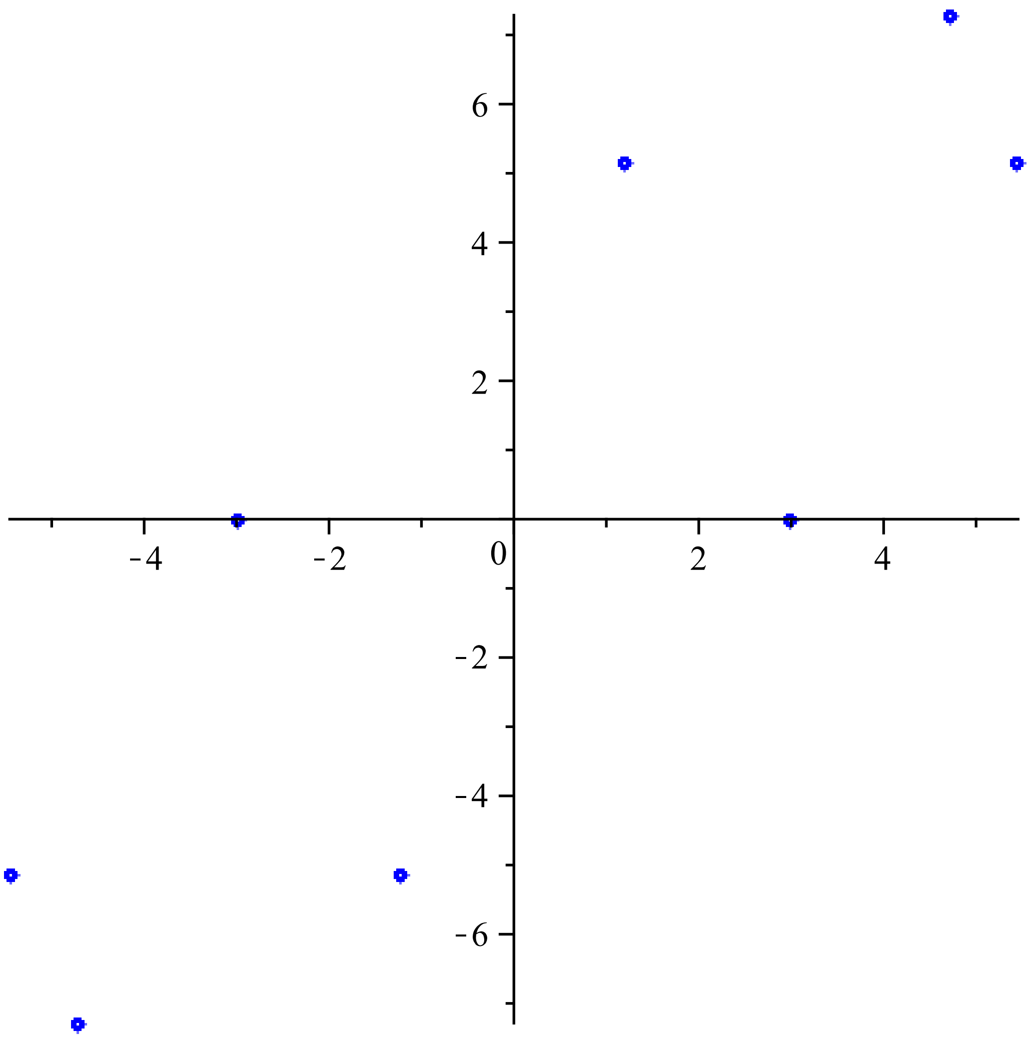
Trial #3:

$$A := \begin{bmatrix} .8 & -.6 \\ .6 & .8 \end{bmatrix}: \quad \text{and} \quad x := \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$



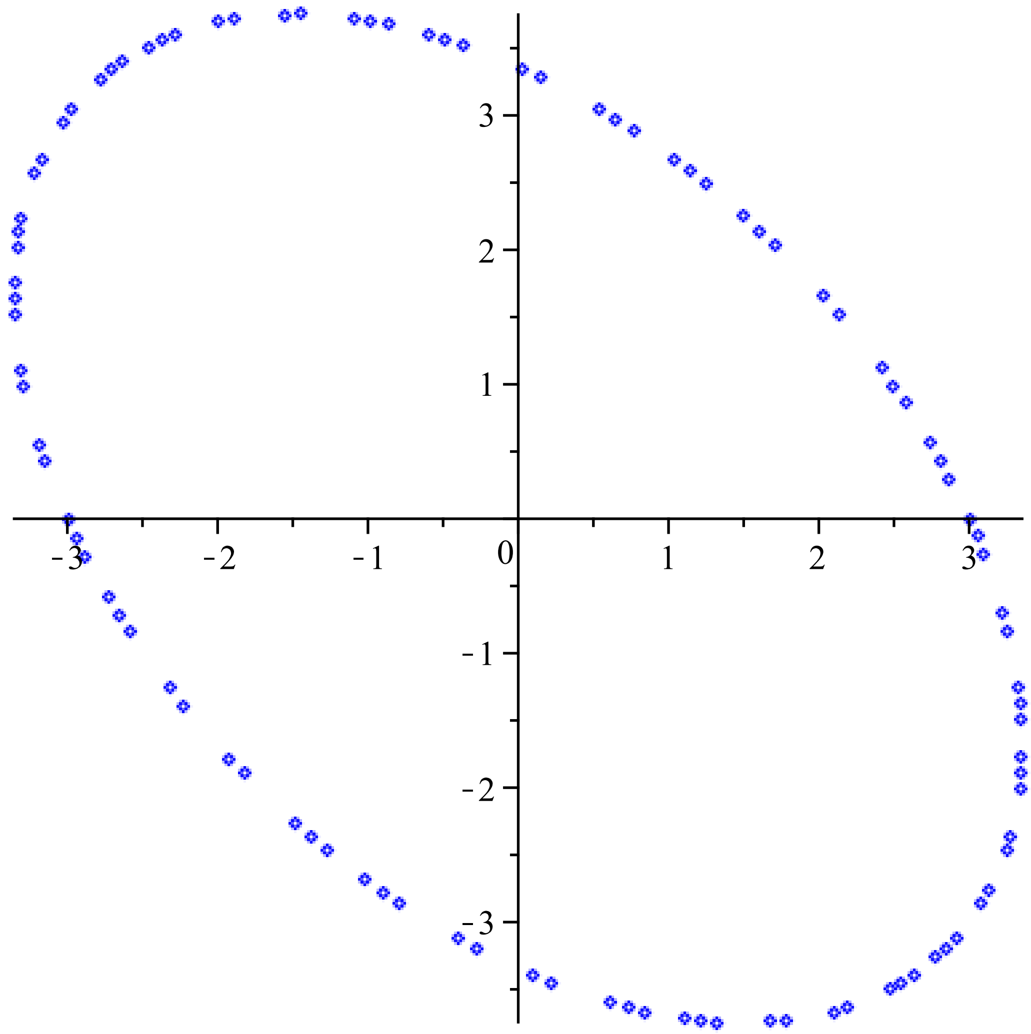
Trial #4:

$$A := \begin{bmatrix} -\frac{2}{7}\sqrt{2} & \frac{5}{7}\sqrt{2} \\ -\frac{17}{14}\sqrt{2} & \frac{9}{7}\sqrt{2} \end{bmatrix} \quad \text{and} \quad x := \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$



Trial #5:

$$A := \begin{bmatrix} 0.5 & -.6 \\ .75 & 1.1 \end{bmatrix} : \qquad \text{and} \qquad x := \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$



Rotation Matrix:

Suppose $C := \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. The eigenvalues of C are found by solving:

$$det(C - \lambda i) = (a - \lambda)^2 + b^2 = 0$$

Thus, $\lambda = a \pm b\,i$

Let $r = |\lambda| = \sqrt{a^2 + b^2}$

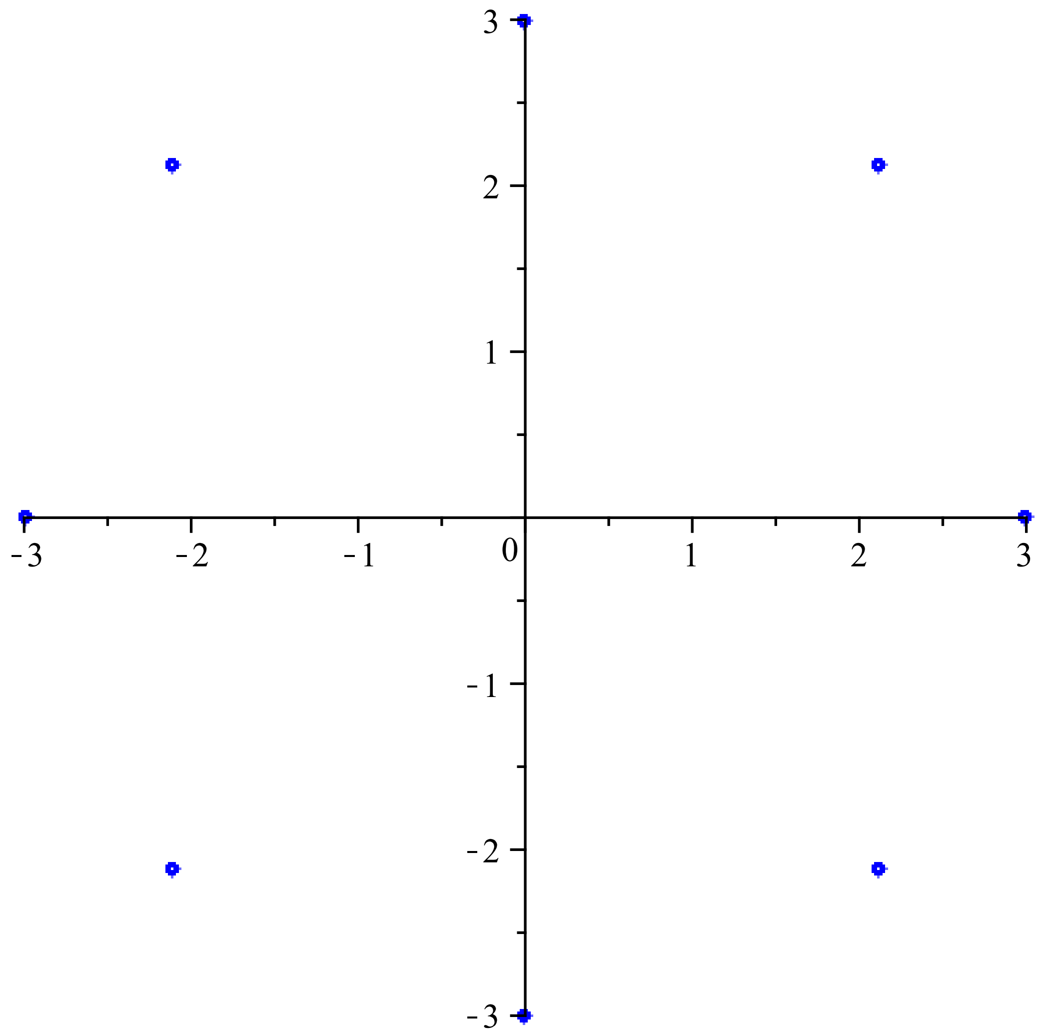
Then C =

$$r \begin{bmatrix} \frac{a}{r} & -\frac{b}{r} \\ \frac{b}{r} & \frac{a}{r} \end{bmatrix} = r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for some θ.

Trial #1--A Rotation Matrix

$$A := \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$



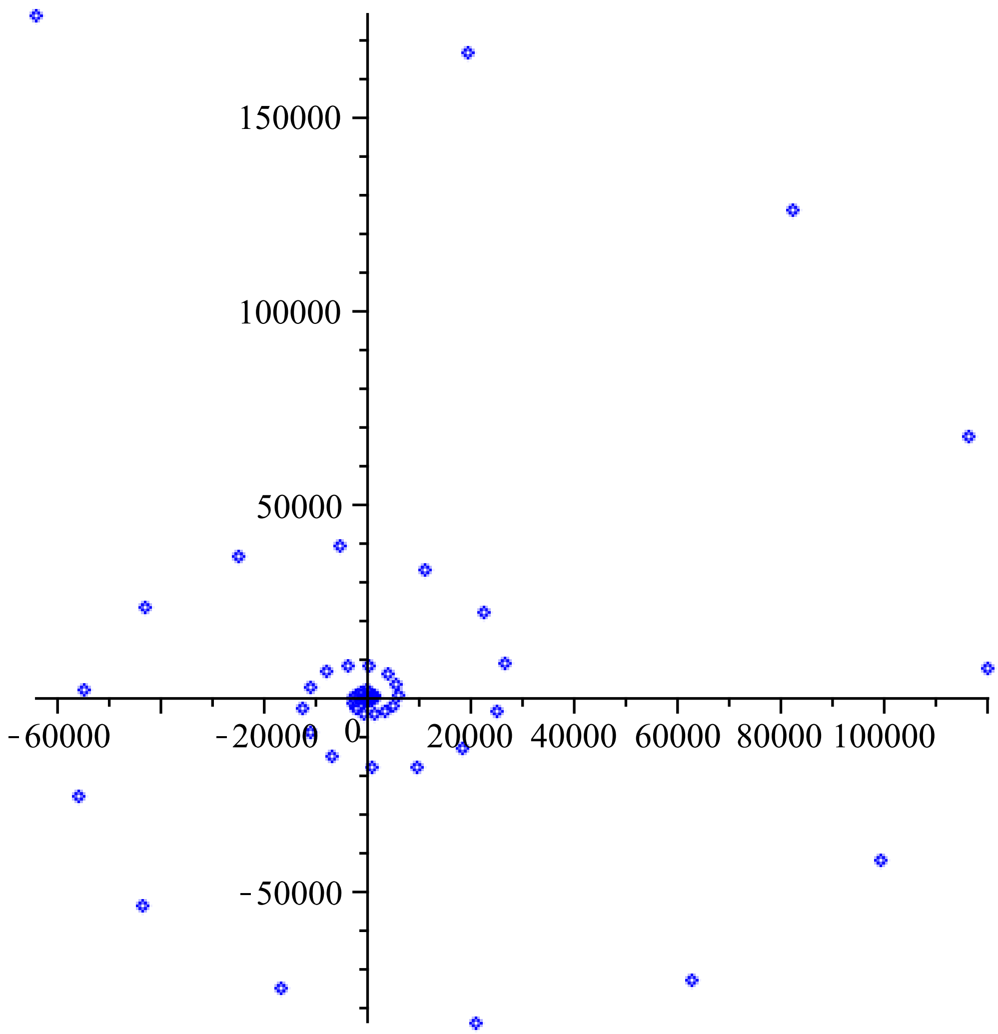
$$\lambda = \frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2} i \quad \text{and} \quad r = |\lambda| = 1$$

$$\theta = \frac{\pi}{4}$$

Thus there are 8 distinct iterates.

Trial #2--A Rotation Matrix

$$A := \begin{bmatrix} 1 & -\frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

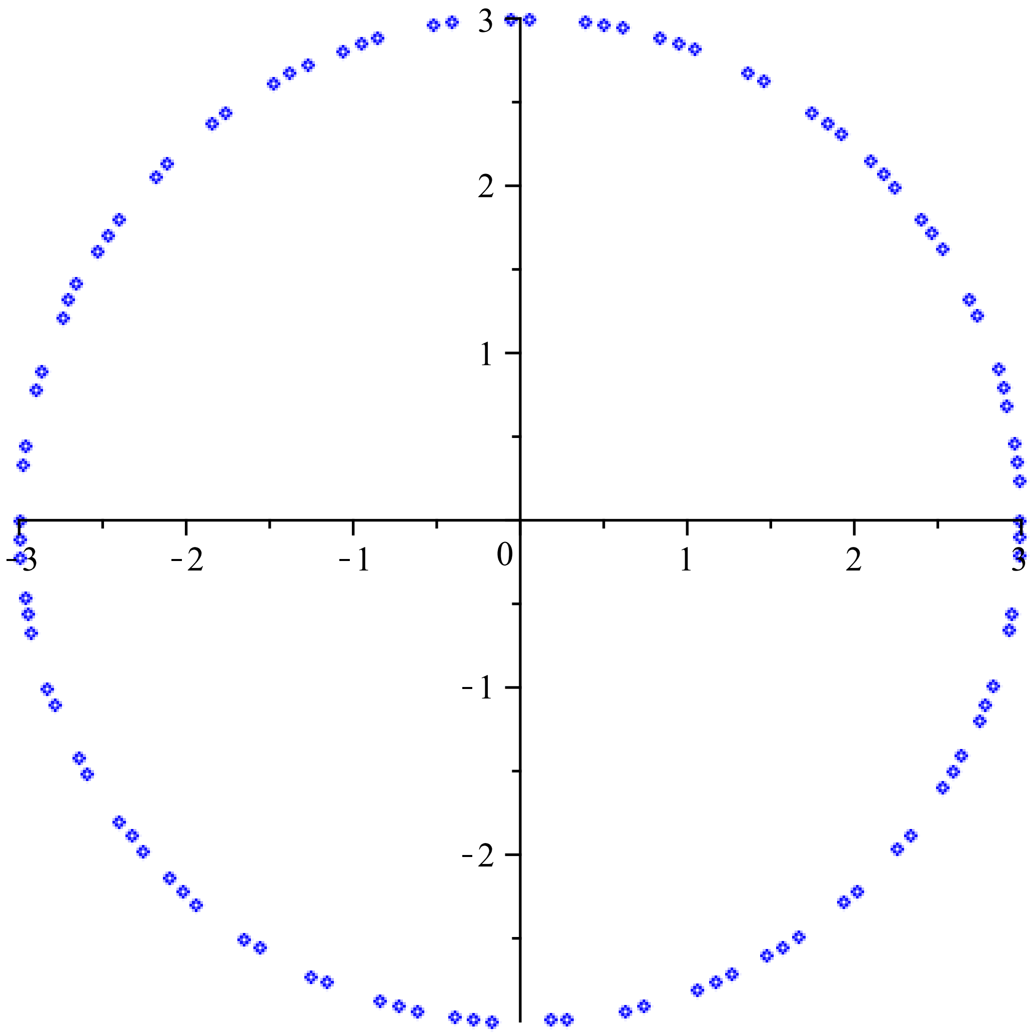


$$\lambda = 1 \pm \frac{1}{2}\,\mathrm{i} \qquad \text{and} \qquad r = |\lambda| = \frac{\sqrt{5}}{2} > 1$$

Thus there will not be repetition in the iterates.

Trial #3--A Rotation Matrix:

$$A := \begin{bmatrix} .8 & -.6 \\ .6 & .8 \end{bmatrix}$$



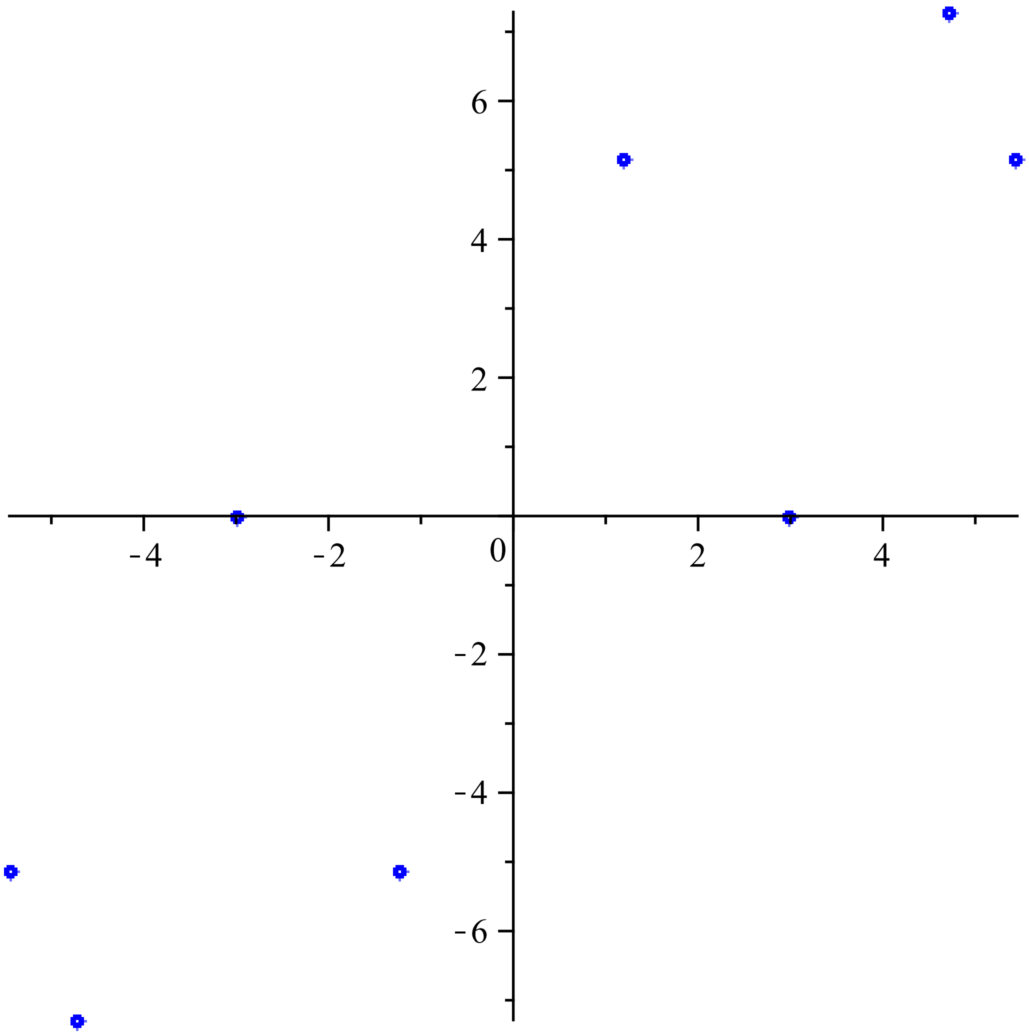
$$\lambda = .8 \pm .6\, i \qquad \text{and} \qquad r = |\lambda| = 1$$

$$\theta = \arctan\left(\frac{.6}{.8}\right)$$

It seems that there will be no repetition in the iterates.

Trial #4:

$$A := \begin{bmatrix} -\frac{2}{7}\sqrt{2} & \frac{5}{7}\sqrt{2} \\ -\frac{17}{14}\sqrt{2} & \frac{9}{7}\sqrt{2} \end{bmatrix}:$$



eigenvals(A)

$$\frac{1}{2}\sqrt{2} - \frac{1}{2}\mathrm{I}\sqrt{2},\ \frac{1}{2}\sqrt{2} + \frac{1}{2}\mathrm{I}\sqrt{2} \tag{1}$$

Explanation:

Theorem:

Let A be a real 2x2 matrix with a complex eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and an associated

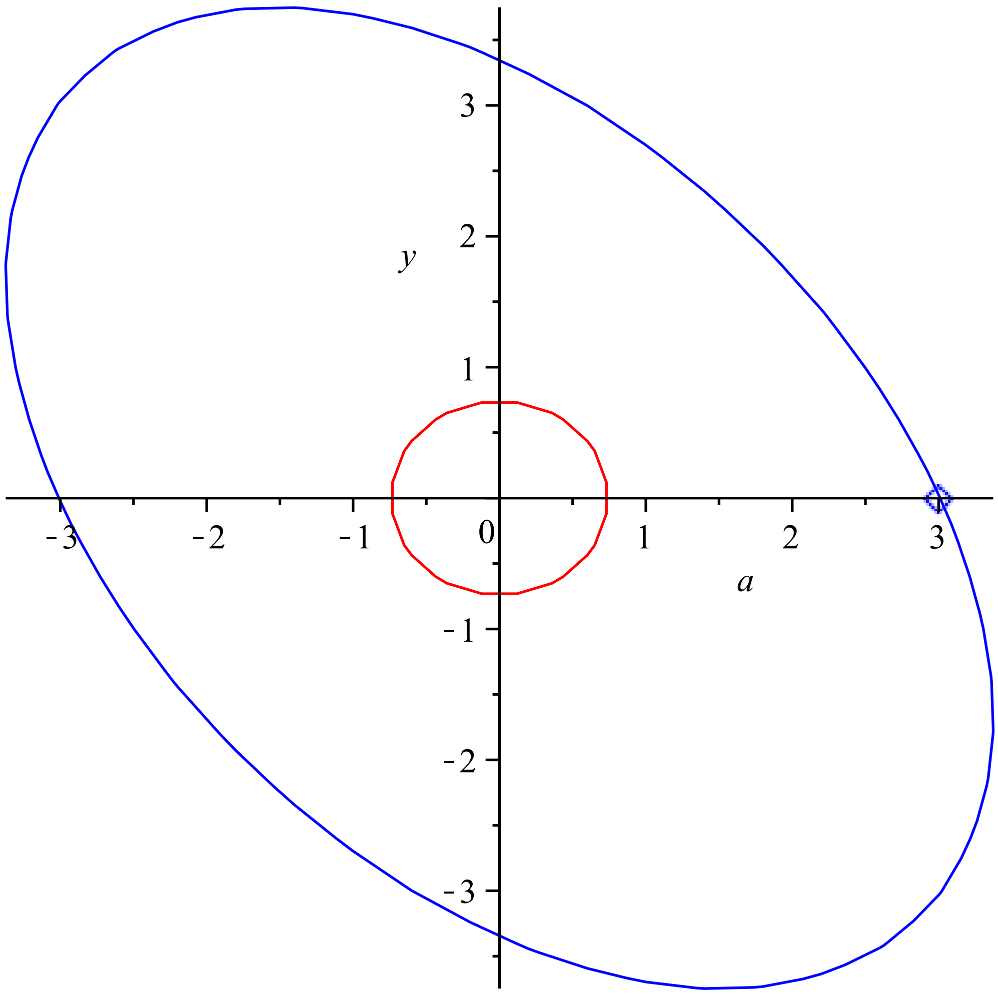
eigenvector $v = \begin{bmatrix} d + ei \\ f + gi \end{bmatrix}$ in C^2. Then

$A = PCP^{-1}$, where $P = [\mathrm{Re}\, v \quad \mathrm{Im}\, v] = \begin{bmatrix} d & e \\ f & g \end{bmatrix}$ and

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

There is the rotation matrix again.

What does this look like?



$$x_0$$

$$x_1 = Ax_0 = PCP^{-1}x_0$$

$$x_2 = A\,x_1 = PCP^{-1}x_1 = PCP^{-1}PCP^{-1}x_0 = PC^2P^{-1}x_0$$

.

.

$$x_n = PC^nP^{-1}x_0$$

Project #2--What is the equation of the ellipse?

Consider A $= PCP^{-1}$ where the eigenvalues of A are $\lambda = a \pm b\,i$ and $|\lambda| = 1$.

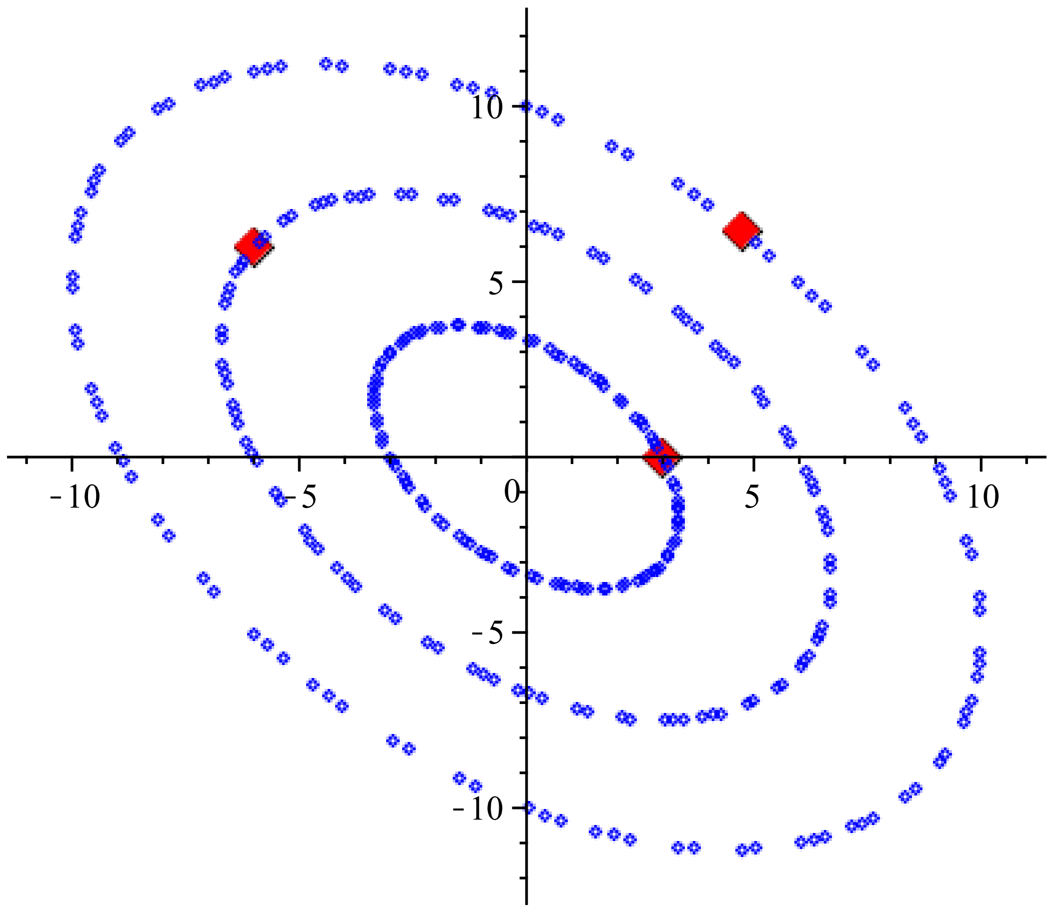
The iterates will all fall on an ellipse.

What is the equation of the ellipse?

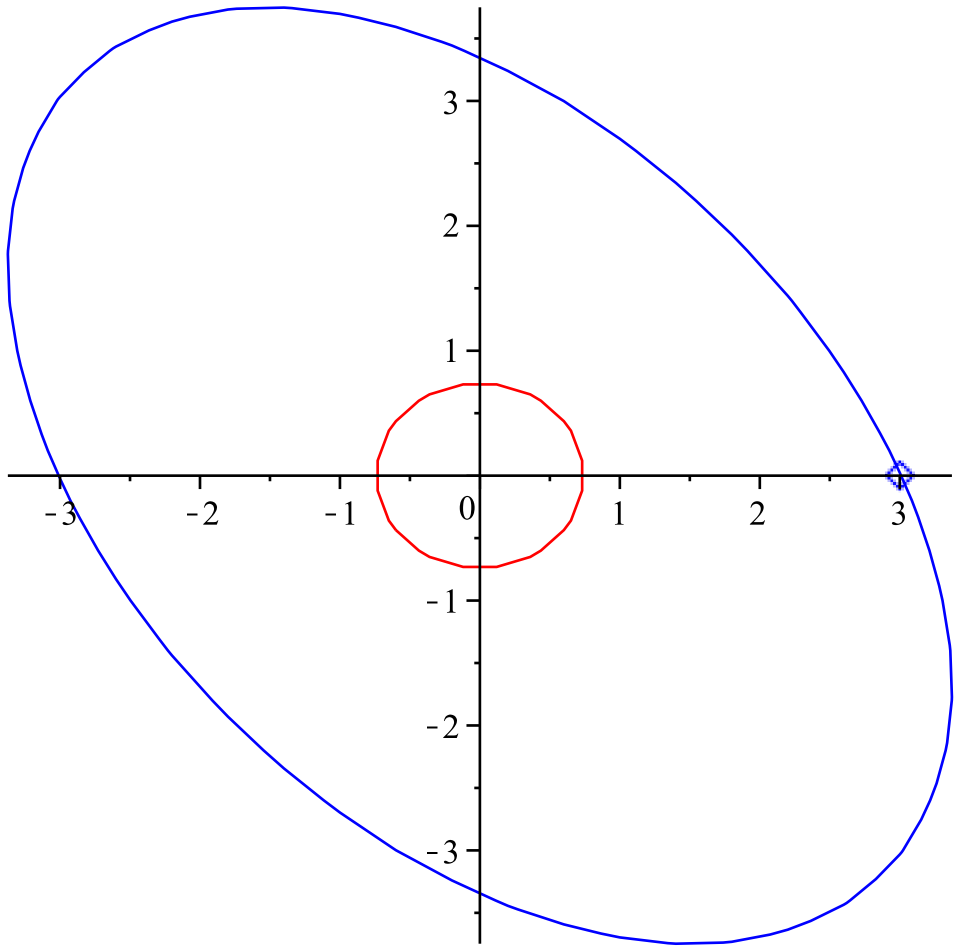
Experimentation

$$A = \begin{bmatrix} 0.5 & -.6 \\ .75 & 1.1 \end{bmatrix}$$

$$x_0 = \left[\begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \end{bmatrix}, \begin{bmatrix} -6 \\ 6 \end{bmatrix} \right]$$



Back to A = $\boldsymbol{PCP}^{-1}$



We know that the circle has radius r = $\left| P^{-1} x_0 \right|$. We know that a point $\begin{bmatrix} X \\ Y \end{bmatrix}$ will be on the ellipse only if

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}$$

for some point $\begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}$ on the circle.

The Equation

$$\left(P_{22}^{2}+P_{21}^{2}\right)X^{2} - 2\left(P_{22}P_{12}+P_{21}P_{11}\right)XY + \left(P_{12}^{2}+P_{11}^{2}\right)Y^{2} = r^{2}det(P)^{2}$$

Construct the symmetric matrix

$$S = \begin{bmatrix} P_{22}^{2}+P_{21}^{2} & -\left(P_{22}P_{12}+P_{21}P_{11}\right) \\ -\left(P_{22}P_{12}+P_{21}P_{11}\right) & P_{12}^{2}+P_{11}^{2} \end{bmatrix}.$$

Our ellipse can be expressed as: $w^{T} S\, w = d$

Where $d = r^{2}\,(det(P))^{2}$ and $w = \begin{bmatrix} X \\ Y \end{bmatrix}$.

Trial #5:

If $A = \begin{bmatrix} 0.5 & -.6 \\ .75 & 1.1 \end{bmatrix}$ and $x_0 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ then:

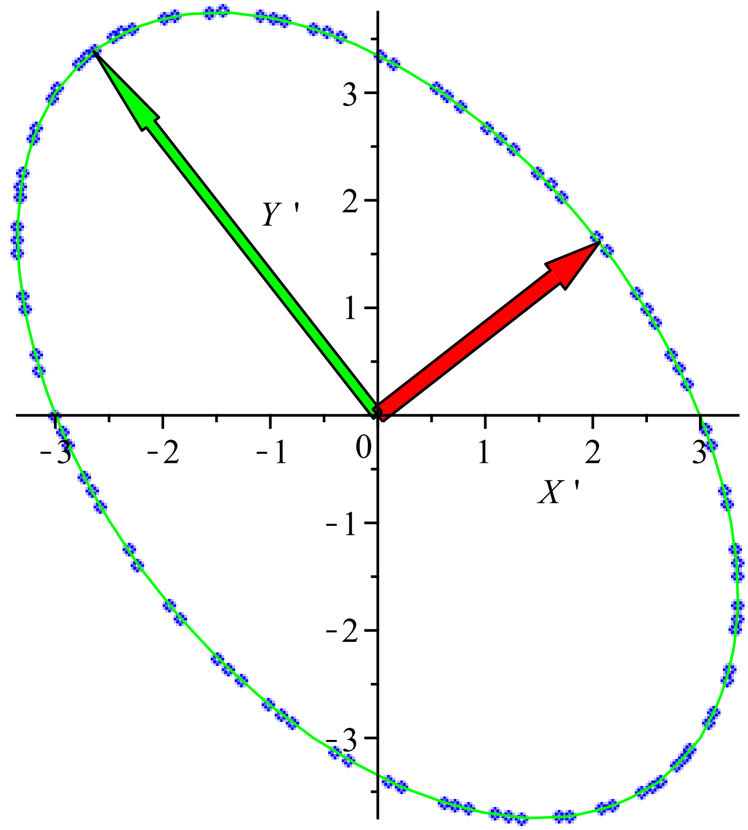
$P = \begin{bmatrix} -2 & -4 \\ 5 & 0 \end{bmatrix}$, $r = \left| P^{-1} x_0 \right| = \frac{3}{4}$, $det(P) = 20$ and $d = r^2 det(P)^2 = 225$.

Then

$$S = \begin{bmatrix} P_{22}^2 + P_{21}^2 & -\left(P_{22}P_{12} + P_{21}P_{11}\right) \\ -\left(P_{22}P_{12} + P_{21}P_{11}\right) & P_{12}^2 + P_{11}^2 \end{bmatrix} = \begin{bmatrix} 25 & 10 \\ 10 & 20 \end{bmatrix}.$$

Equation of the ellipse: $25\,X^2 + 20\,XY + 20\,Y^2 = 225$

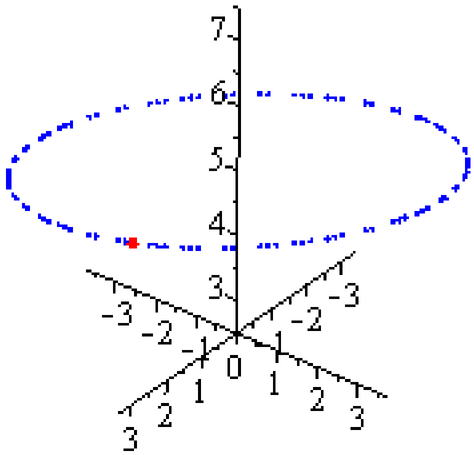
The Graph with Axes.

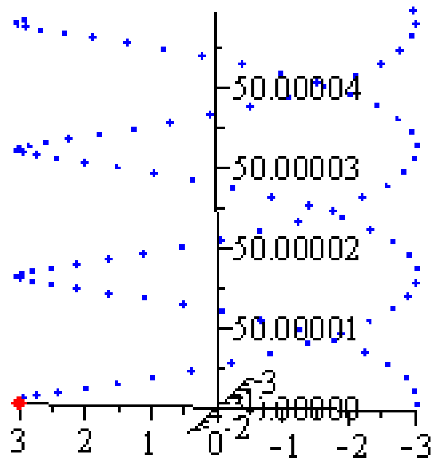


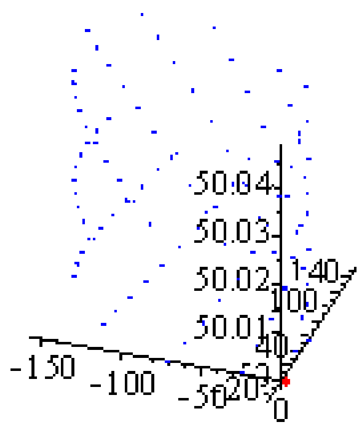
This lead to another project discussing orthogonal diagonalization and major and minor axes of the ellipse.

And the Latest:

Three dimensions







In Closing:

- Students used the computer algebra system Maple to experiment with ideas, to visualize results, and ultimately to form hypotheses that they proved.

- The four step method was easy for students to understand and to mimic. Coding was never the problem. The mathematics was.

- Maple made the experimentation easy. Ideas were formed and Maple was used for the brute force calculations. Ideas were quickly discarded too when results were not as expected.

- Without the power of Maple, it is difficult to imagine these students forming the hypotheses they did. There was simply too much calculation to do by hand.

- Make sure you are ready to invest a lot of time before embarking on these types of student projects.

Conclusion: It is worth the time.