Transforming Linear Algebra with GeoGebra

JMM 2014

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NSF - TUES Grant 1141045:
Transforming Linear Algebra Education with GeoGebra

Note:

Copies of the draft applets presented at the JMM 2014 and the software to run them are at this location:

https://www.dropbox.com/sh/gff2xggpe815rgg/sy-KbOK_oy

- **Please look at READ ME file to run the software**

If you are interested in getting these applets and the others being developed, have comments on improving them, and/or suggestions on topics you want to see covered, please contact me at my email address:

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Transforming Linear Algebra Education with GeoGebra Applets

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Alverno College

GOAL:

Deepen understanding of linear algebra concepts

- Actively engage the student in the *geometric*, *algebraic*, and *numeric* perspectives of the concept
- Through interactive use of 2D and 3D applets
- Enhancing problem solving skills

Each ***Applet Package*** includes
- instructional support and
- a STEM application.

Vector Package

12 Linear Algebra Packages

GeoGebra 5.0 Beta

	Sequence 1: **Vectors**	***Sequence 2:*** **Linear Systems of Equations**	***Sequence 3:*** **Matrices**	***Sequence 4:*** **Vector Spaces**
Level 1	***Vector Package*** Points vs Vectors Addition/Subtraction Scalar Product Commutative & Associative Properties Linear Combinations	***Views of Linear Systems Package*** Matrix View Row View Column View	***Matrix Package*** Matrix Operations Matrix Properties Linear Independence Transpose	***Vector Spaces Package*** Null Spaces Row Spaces Column Spaces Linear Transformations
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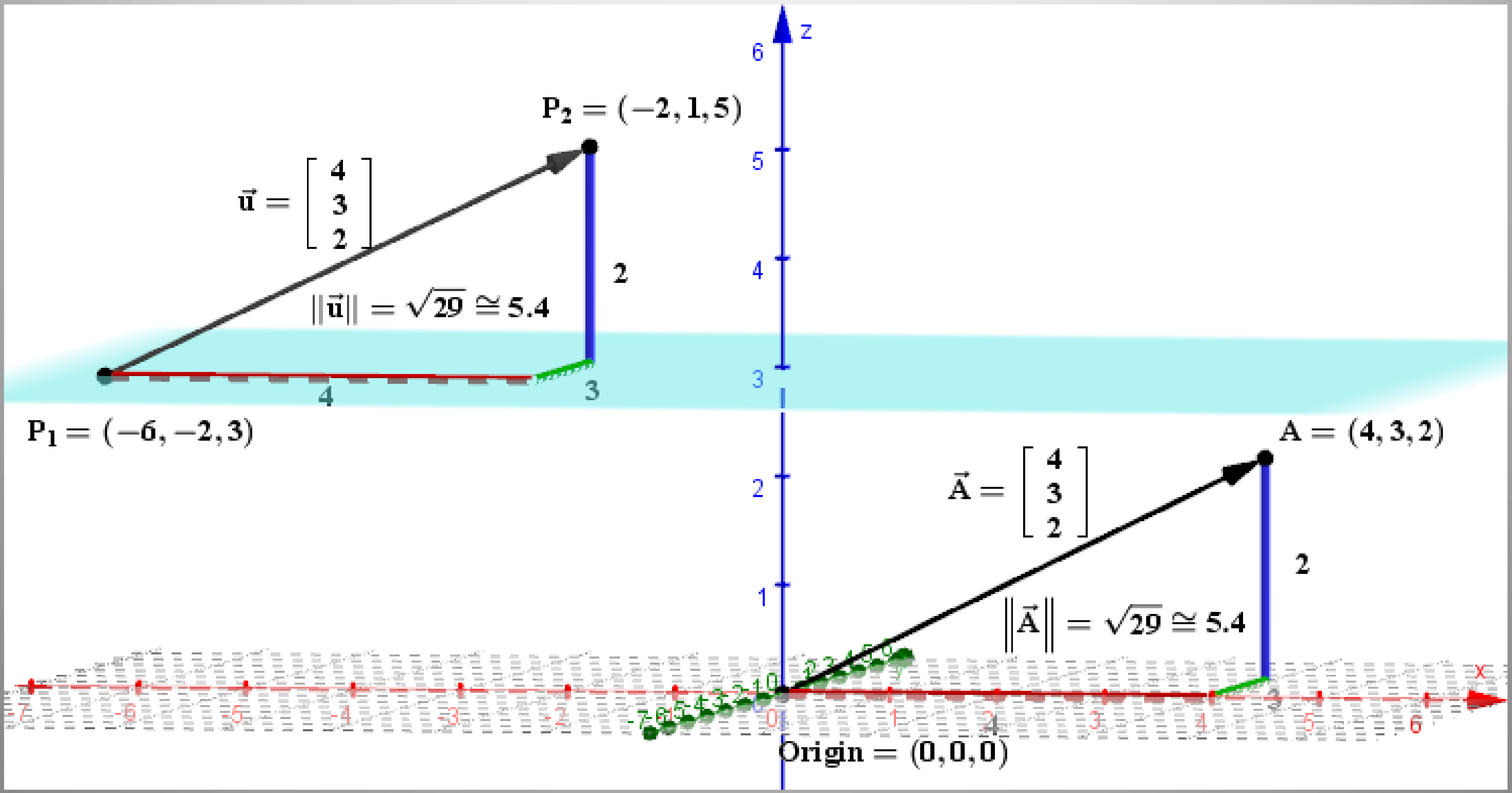
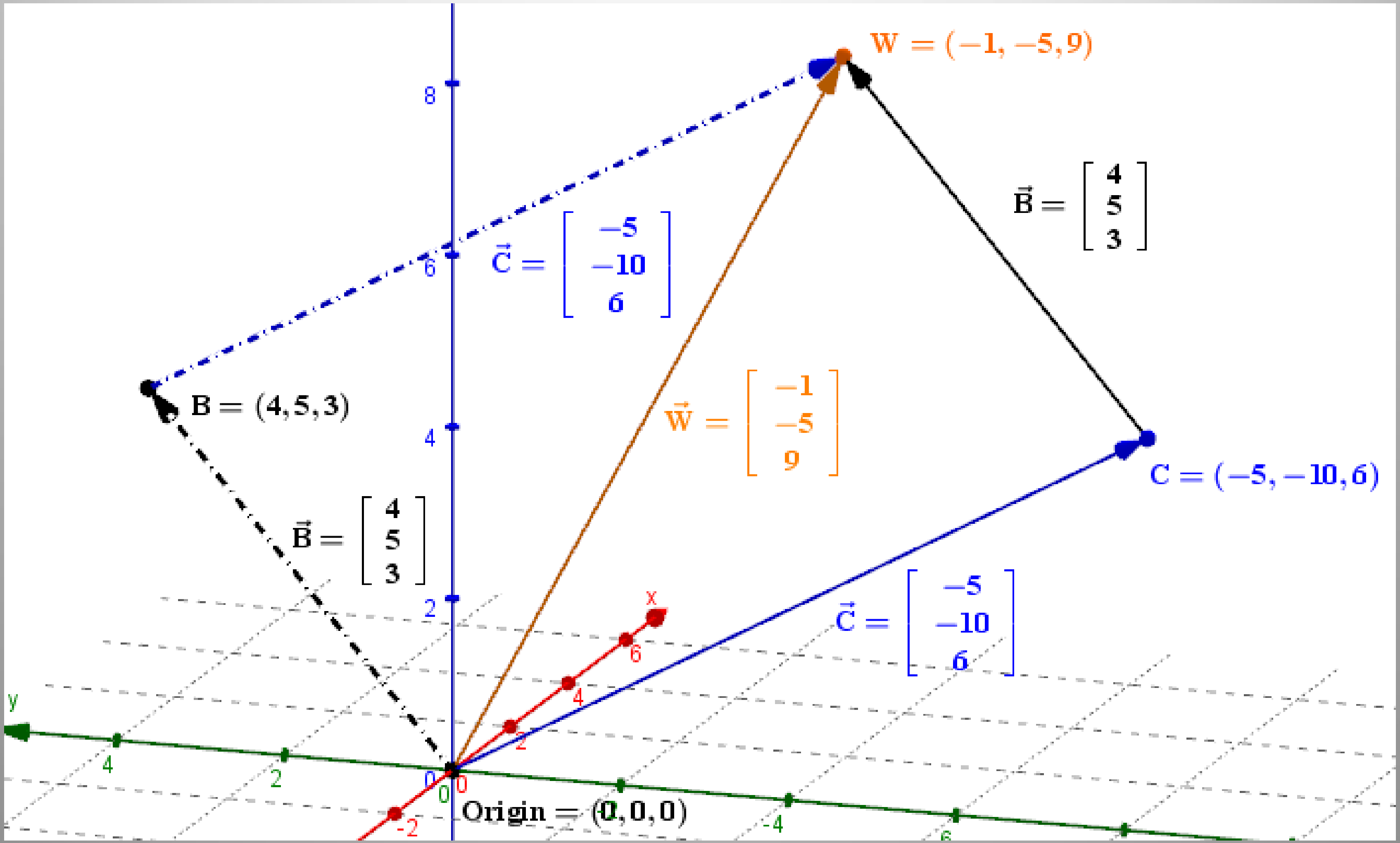
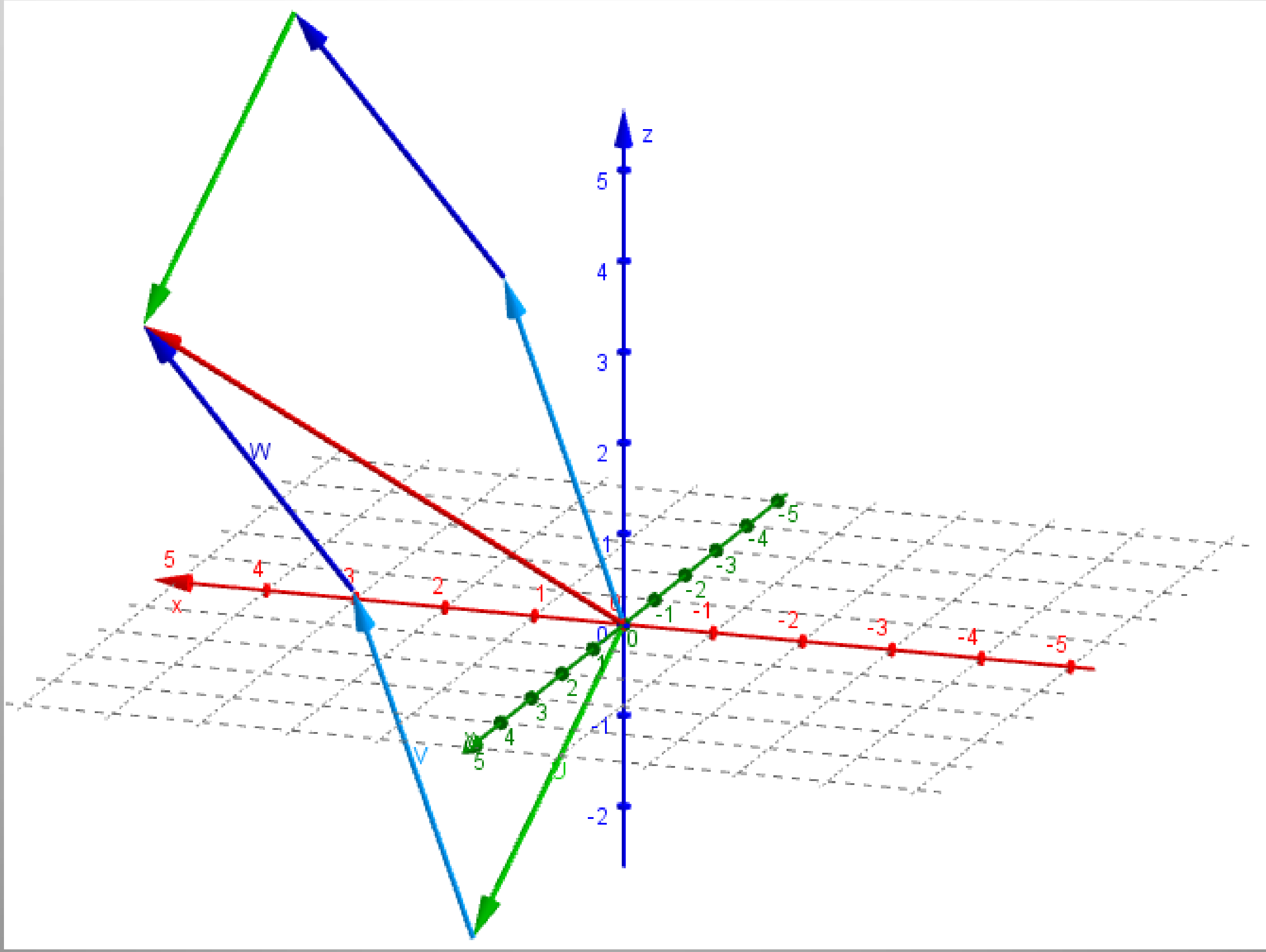
Vectors Sequence 1 (Fundamental Level 1) - Vector Package Sample



Adding Vectors, showing Commutative Property in 3D

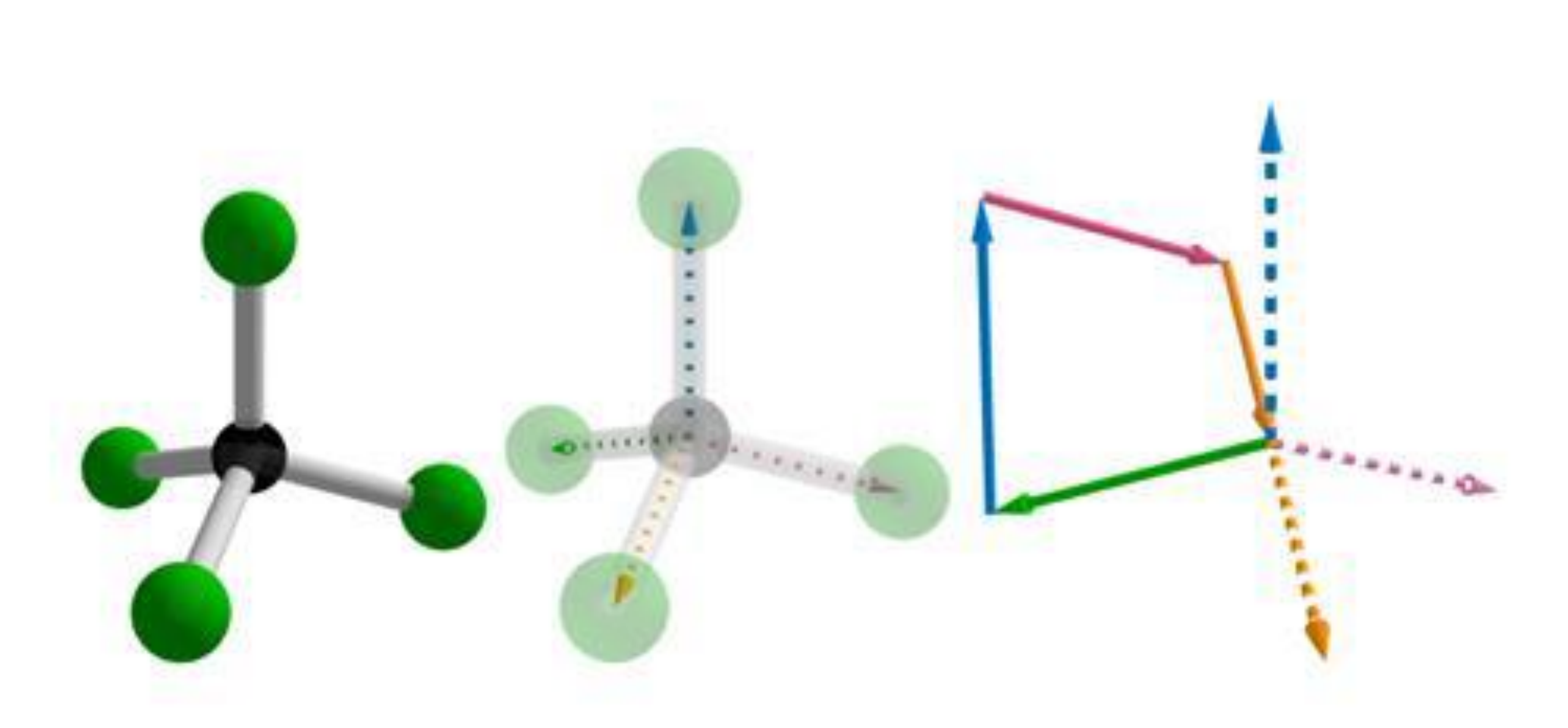


Associativity of Vectors – 3D



STEM – Application

3D Application to CCl_4: Using the Addition of Vectors



Coordinate Grid Package

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Vector Sequence 1 (Intermediate Level 2) - Coordinate Grid Package -- Sample

Overview:

Coordinate Grid Package 2D - (S1L2)

– Exploring new coordinate systems in R^2

This applet involves change of coordinate systems in R^2 and visualization of linear combinations of vectors.

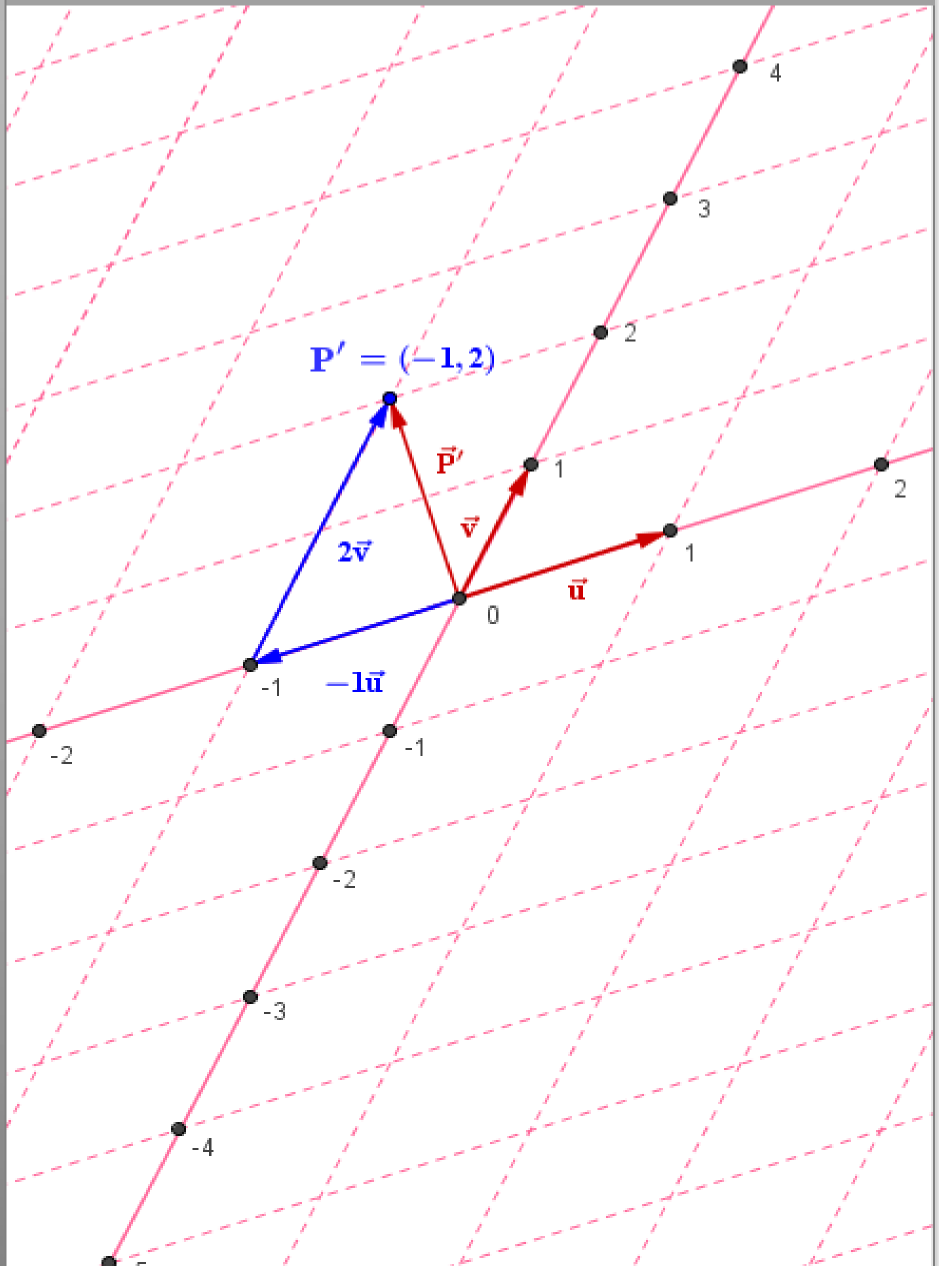
Student Learning Goals:

In this activity, students will:

- Explore the geometric, algebraic, and numeric meaning of a linear combination of vectors.
- Visualize a coordinate system in two dimensions based on two arbitrary linearly independent vectors.
- Identify the basis vectors of any two-dimensional coordinate system.
- Express coordinates of any point in the plane in terms of standard and non-standard coordinate systems.
- Develop facility with geometric imagination about alternate coordinate systems.
- Understand the geometric and algebraic similarities and differences of describing a point or a vector in different coordinate systems.
- Develop facility to write vectors as linear combinations of other vectors no matter what the coordinate system.
- Make conjectures about describing points and basis vectors in different coordinate systems and test them.

Coordinate Grid Package - Change of Basis

Overview | Importance | Coordinate Grids | Application



UV-Coordinate Grids

☑ On the XY-Grid, input linearly independent vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$:

For sliders, uncheck: By components.

☑ By components.

$\vec{\mathbf{u}} = [$ 3 , 1 $]$

$\vec{\mathbf{v}} = [$ 1 , 2 $]$

$\vec{\mathbf{u}} = [3, 1]$ $\vec{\mathbf{v}} = [1, 2]$ The basis of a new coordinate system.

☐ Show angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. ☐ *

☑ Define a UV-Coordinate Grid based on these vectors. ☐ *

Turn off the XY-Coordinate Grid. ☐

Refer to the point being analyzed on the UV-Grid as $\mathbf{P'}$.

☑ Consider point P, referenced as $\mathbf{P'}$.

Find the coordinates of this point, $\mathbf{P'}$. ☑ →

Using these sliders, move along the

U-axis to find the multiple of $\vec{\mathbf{u}}$ needed to reach the point..

$C_u = -1$ $-1\vec{\mathbf{u}}$

V-axis to find the multiple of $\vec{\mathbf{v}}$ needed to reach the point.

$C_v = 2$ $2\vec{\mathbf{v}}$

Check your results: ☑ along the U-axis we have: $-1\vec{\mathbf{u}}$

☑ along the V-axis we have: $2\vec{\mathbf{v}}$

☑ On this UV-Grid, draw the vector $\vec{\mathbf{P}}'$ from the origin to $\mathbf{P'}$.

☑ Algebraically, write $\vec{\mathbf{P}}'$ as a linear combination of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

$$\vec{\mathbf{P}}' = -1\vec{\mathbf{u}} + 2\vec{\mathbf{v}}$$

☑ The coordinates of $\mathbf{P'}$ on the UV-Grid are: $\mathbf{P'} = (-1, 2)$

☑ State $\vec{\mathbf{P}}'$ using vector notation: $\vec{\mathbf{P}}' = [-1, 2]$

What do the coordinates of $\mathbf{P'} = (\mathbf{u}, \mathbf{v})$ on the UV-Grid correspond to in the linear combination? ☑

The first component $\mathbf{u} = -1$ is the coefficient of .

The second component $\mathbf{v} = 2$ is the coefficient of $\vec{\mathbf{v}}$.

☐ Try other points. ☐ Clear Screen

Linear Systems Package

12 Linear Algebra Packages

GeoGebra 5.0 Beta

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S2L2 – Solving Linear Systems Package

Example: Find the intersection of the planes

$x + 2y - z = 3$ and $2x + 3y + z = 1$.

The normal to the first plane is **[1, 2, -1]**
The normal to the second plane is **[2, 3, 1]**

What do these two normals tell us about whether these planes intersect?

The points that lie on the intersection of the two planes correspond to the points in the solution set of the system

$$x + 2y - z = 3$$
$$2x + 3y + z = 1$$

We can solve using the Gauss-Jordan method.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 3 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 5 & -7 \\ 0 & 1 & -3 & 5 \end{array}\right] \Rightarrow$$

$$\begin{aligned} x + 5z &= -7 \\ y - 3z &= 5 \end{aligned} \Rightarrow \begin{aligned} x &= -7 - 5z \\ y &= 5 + 3z \end{aligned} \Rightarrow \begin{aligned} x &= -7 - 5t \\ y &= 5 + 3t \\ -\infty &< t < \infty \end{aligned}$$

⇒ In vector form: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix}$

But what is this geometrically?

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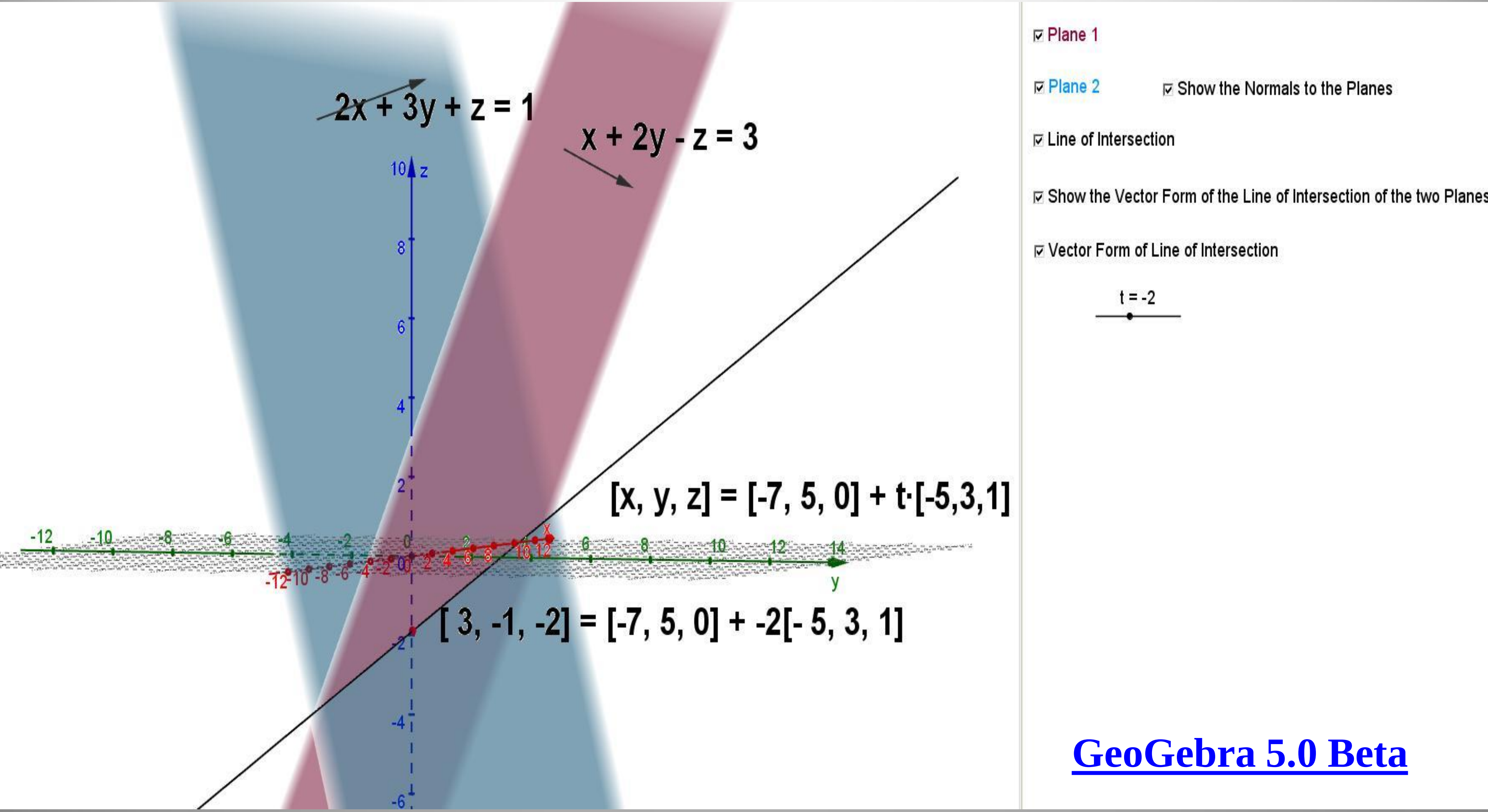
Solution



[GeoGebra 5.0 Beta]

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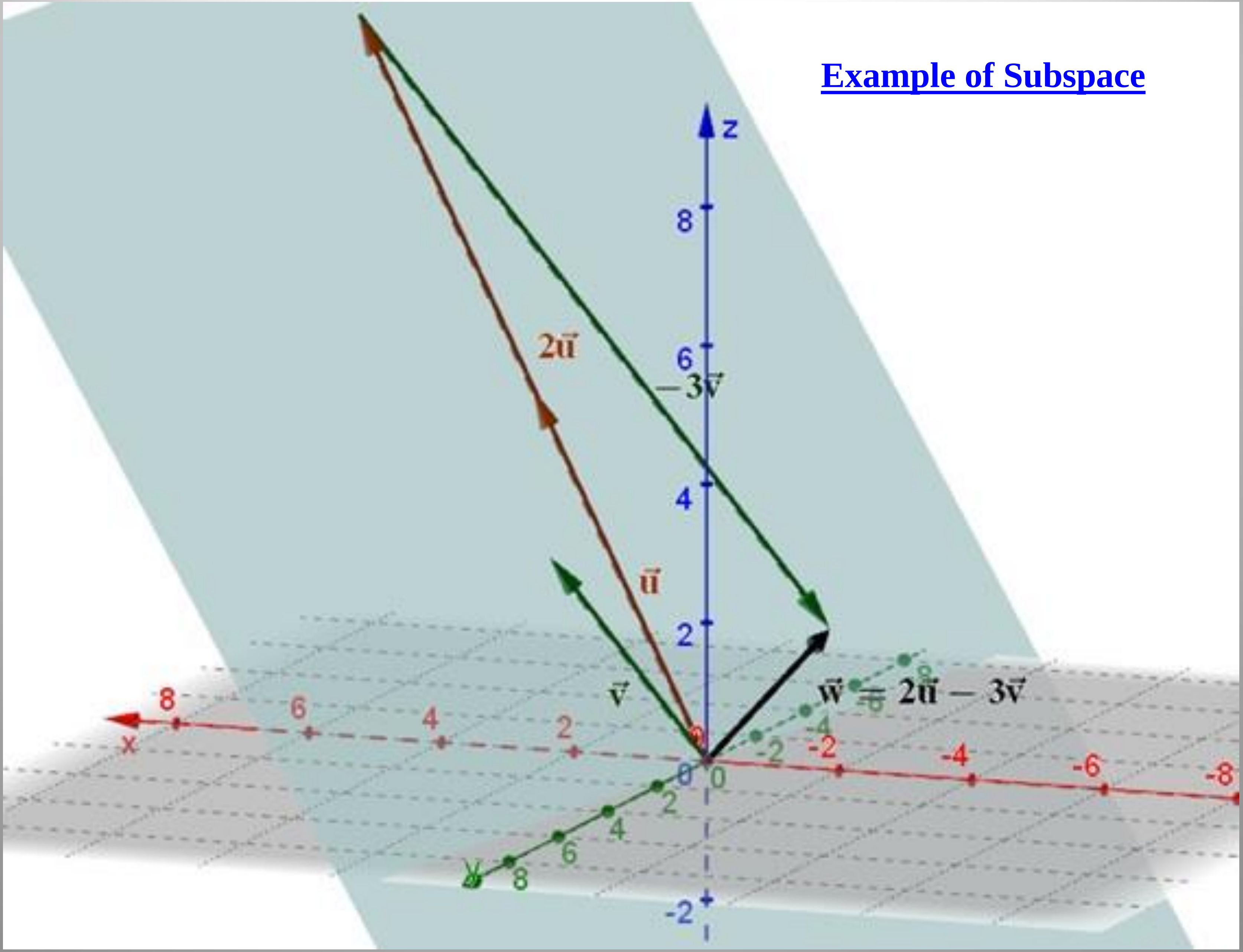
Example of Subspace



Questions ?

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Slide not in JMM 2014 Presentation but possibly of interest