Tridiagonal Matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} \qquad \frac{2}{3} = \frac{10}{15}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} = \begin{bmatrix} 37 & 54 \\ 81 & 112 \end{bmatrix} \quad \frac{2}{3} = \frac{54}{81}$$

It happened twice, that looks like a theorem

All functions of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ have the

same off-diagonal ratio $\dfrac{b}{c}$

First noticed and proved by induction

for 2×2: Peter Larcombe

The result extends to all **tridiagonal matrices**

Linear algebra proof

Choose a diagonal matrix D so that

$DAD^{-1} = \begin{bmatrix} a & d_1 b/d_2 \\ d_2 c/d_1 & d \end{bmatrix}$ is **symmetric**

Then all its powers are symmetric

Now undo the similarity

$$A^n = D^{-1}(DAD^{-1})^n D$$

has the same off-diagonal ratios for all n

True for any function of A including A^{-1} and

$$xy^{\mathsf{T}} = (\text{eigenvector})(\text{left eigenvector})^{\mathsf{T}}$$

Ira Gessel's "combinatorial proof"

Compare the $1,1$ entries of $AA^n = A^nA$

$$\begin{bmatrix} a_1 & b_1 & \\ c_1 & a_2 & b_2 \\ \bullet & c_2 & \bullet \end{bmatrix} \begin{bmatrix} A_1 & B_1 & \\ C_1 & A_2 & B_2 \\ & C_2 & \bullet \end{bmatrix} = \begin{bmatrix} a_1A_1 + b_1C_1 & \bullet \\ \bullet & \bullet \\ & \end{bmatrix}$$

$$\begin{aligned} a_1A_1 + b_1C_1 &= A_1a_1 + B_1c_1 \\ b_1/c_1 &= B_1/C_1 \qquad \text{QED} \end{aligned}$$

Extension to 2nd order differential equations

"Tridiagonal matrices can be
2nd order difference operators"

$$\left(a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x)\right) u(x)$$

can be symmetrized by changing
from $u(x)$ to $v(x) = D(x)u(x)$
The symmetric form is

$$\frac{d}{dx}\left(A(x)\frac{d}{dx}\right) v(x) + C(x)v(x)$$

David Bindel : apply to eigenfunctions
Partial Differential Equations
Symmetrize $au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu$
In $2D$ it helps to have **constant coefficients**

Inverses of tridiagonal matrices

$$A^{-1} = \begin{bmatrix} \ddots & uv^{\mathrm{T}} \\ wz^{\mathrm{T}} & \ddots \end{bmatrix}$$

"rank 1 above and on the diagonal"
"rank 1 below and on the diagonal"

$$A^{-1} = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 6 & 10 \\ 3 & 18 & 4 \end{bmatrix}$$

Then A is tridiagonal:

$$A_{13} = \frac{\text{cofactor}}{\text{determinant}} = 0$$

$$A_{31} = \frac{\text{cofactor}}{\text{determinant}} = 0$$

Completion of tridiagonal matrices

$$\textbf{Start} \qquad A_0 = \begin{bmatrix} 6 & 2 & x \\ 2 & 8 & 4 \\ x & 4 & 3 \end{bmatrix}$$

Which number x **maximizes the determinant** ?

Then x also maximizes the "entropy"

Answer : $\boldsymbol{x = 1}$ This gives rank 1 away from the diagonal

The inverse of the completed matrix is tridiagonal

$$\frac{d}{dx}(\det A) = \frac{\text{cofactor}}{\text{determinant}} = 0 \text{ for best } x$$

Tough decisions on the notation !

1. Do I have to use *ln* for log ?
2. Variable of integration on page 1 ?

$$\text{Solution to } \frac{dy}{dt} = f(t) \qquad y(t) = \int_0^t f(s)ds + C$$

$$\int_0^t f(\tau)\, d\tau \qquad \int_0^t f(T)\, dT \qquad \int_0^t f(t)\, dt$$

3. Can solutions and computer codes go on `math.mit.edu/dela` ?