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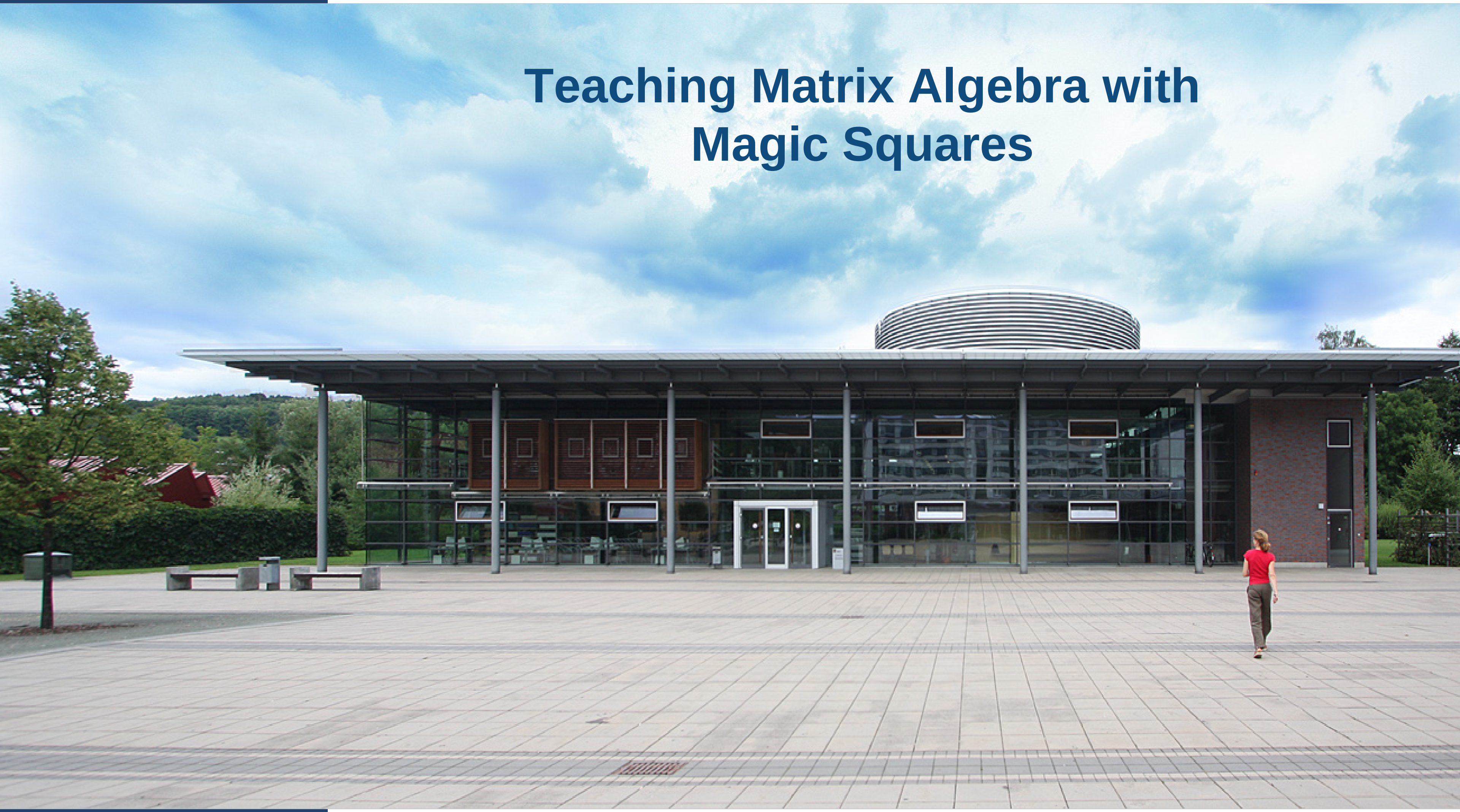
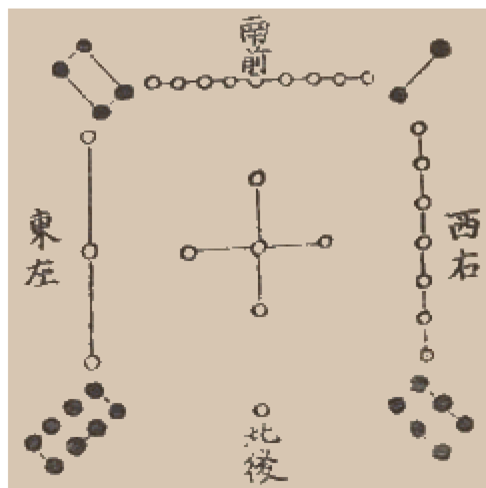
Teaching Matrix Algebra with Magic Squares



In recreational mathematics, a [semi-] magic square of order n is an arrangement of n^2 numbers, usually distinct integers, in a square, such that the n numbers in all rows, all columns, and both diagonals sum to the same constant. A normal magic square contains the integers from 1 to n^2.

History I

Lo-Shu, China, 650 BC



$$L = \begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix}$$

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History II

Melencolia I
by
Albrecht
Dürer,
1514

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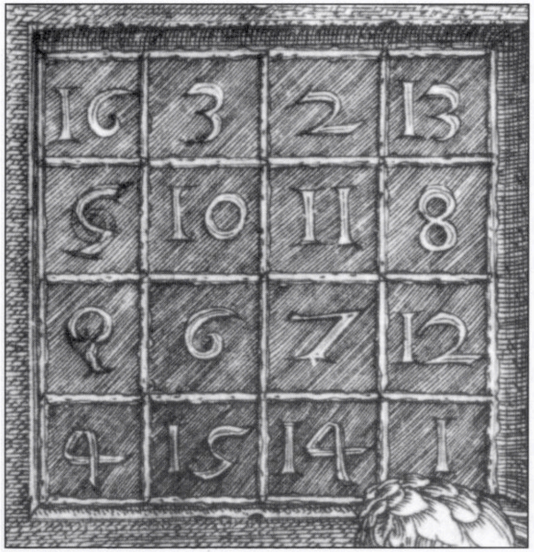
History II



$$\boldsymbol{D} = \begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix}$$

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History III

Basílica de la Sagrada Família, Barcelona, Spain (1882-2026)

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History III

$$\boldsymbol{S} = \begin{pmatrix} 1 & 14 & 14 & 4 \\ 11 & 7 & 6 & 9 \\ 8 & 10 & 10 & 5 \\ 13 & 2 & 3 & 15 \end{pmatrix}$$

Fachada de la Pasión (by Josep Subirachs):
El beso de Judas

Is *L* really magic? (matrix operations and trace I)

$$\boldsymbol{L}=\begin{pmatrix}4 & 9 & 2\\ 3 & 5 & 7\\ 8 & 1 & 6\end{pmatrix};\quad \mathbf{1}=\begin{pmatrix}1\\ 1\\ 1\end{pmatrix};\quad \boldsymbol{F}=\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}$$

$$\boldsymbol{L}\mathbf{1}=\begin{pmatrix}15\\ 15\\ 15\end{pmatrix} \qquad \mathbf{1}'\boldsymbol{L}=\begin{pmatrix}15 & 15 & 15\end{pmatrix}$$

$$\operatorname{tr}(\boldsymbol{L})=15 \qquad \operatorname{tr}(\boldsymbol{FL})=\begin{vmatrix}8 & 1 & 6\\ 3 & 5 & 7\\ 4 & 9 & 2\end{vmatrix}=15$$

Is *D* really magic? (matrix operations and trace II)

$$\boldsymbol{D} = \begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix}; \quad \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}; \quad \boldsymbol{F} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{D}\mathbf{1} = \begin{pmatrix} 34 \\ 34 \\ 34 \\ 34 \end{pmatrix}; \quad \mathbf{1}'\boldsymbol{D} = \begin{pmatrix} 34 & 34 & 34 & 34 \end{pmatrix}$$

$$\operatorname{tr}(\boldsymbol{D}) = 34; \quad \operatorname{tr}(\boldsymbol{FD}) = \begin{pmatrix} 4 & 15 & 14 & 1 \\ 9 & 6 & 7 & 12 \\ 5 & 10 & 11 & 8 \\ 16 & 3 & 2 & 13 \end{pmatrix} = 34$$

Is *S* really magic? (matrix operations and trace III)

$$\boldsymbol{S}=\begin{pmatrix}1 & 14 & 14 & 4\\ 11 & 7 & 6 & 9\\ 8 & 10 & 10 & 5\\ 13 & 2 & 3 & 15\end{pmatrix};\quad \boldsymbol{1}=\begin{pmatrix}1\\1\\1\\1\end{pmatrix};\quad \boldsymbol{F}=\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}$$

$$\boldsymbol{S1}=\begin{pmatrix}33\\33\\33\\33\end{pmatrix};\quad \boldsymbol{1}'\boldsymbol{S}=\begin{pmatrix}33 & 33 & 33 & 33\end{pmatrix}$$

$$\operatorname{tr}(\boldsymbol{S})=33;\quad \operatorname{tr}(\boldsymbol{FS})=33$$

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Ax = λx (eigenvalues and -vectors)

$\boldsymbol{M}$, $n \times n$ matrix denoting magic square

$\mathbf{1}$, $n \times 1$ vector of ones

s, magic sum of $\boldsymbol{M}$

eigenvector

$$\boldsymbol{M}\mathbf{1} = s\mathbf{1}$$

eigenvalue

Examples: ***L***, ***D*** , ***S***

EIGENVALUES(L) = [**15**, √24·*i*, -√24·*i*]

EIGENVALUES(D) = [**34**, 0, 8, -8]

EIGENVALUES(S) = [**33**, ≈0.8445, ≈-7.734, ≈6.890]

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Parameterization of 3x3 magic squares I

$$\underset{3\times 3}{\boldsymbol{M}} = s\boldsymbol{G} + \boldsymbol{N}; \quad \boldsymbol{G} = \tfrac{1}{3}\mathbf{1}\mathbf{1}' = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}; \quad \boldsymbol{N} = \begin{pmatrix} \alpha+\beta & -2\alpha & \alpha-\beta \\ -2\beta & 0 & 2\beta \\ -\alpha+\beta & 2\alpha & -\alpha-\beta \end{pmatrix}$$

Example: ***L***

```
#5:   M := s·G + N

             [ 4  9  2 ]
#6:   L :=   [ 3  5  7 ]
             [ 8  1  6 ]

#7:   M = L

#8:   SOLVE(M = L, [s, α, β])

#9:                     s = 15 ∧ α = -2 ∧ β = 1
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Parameterization of 3x3 magic squares II

$$\underset{3\times3}{\mathbf{M}} = s\mathbf{G} + \mathbf{N}; \quad \mathbf{G} = \tfrac{1}{3}\mathbf{1}\mathbf{1}' = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}; \quad \mathbf{N} = \begin{pmatrix} \alpha+\beta & -2\alpha & \alpha-\beta \\ -2\beta & 0 & 2\beta \\ -\alpha+\beta & 2\alpha & -\alpha-\beta \end{pmatrix}$$

	trace	det	rank	eigenvalues
G	1	0	1	[0, 0, 1]
N	0	0	2	[0, √(12αβ), -√(12αβ)]
M	s	- s·12αβ	3	[s, √(12αβ), -√(12αβ)]
L	15	360	3	[15, √24·*i*, -√24·*i*]

The 3 examples again (trace, det, rank, eigenvalues)

	trace	det	rank	eigenvalues
L	15	360	3	[15, $\sqrt{24}\cdot i$, $-\sqrt{24}\cdot i$]
D	34	0	3	[34, 0, 8, -8]
S	33	-1485	4	[33, ≈0.8445, ≈-7.734, ≈6.890]

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Inverses I

L is a nonsingular matrix

$$\boldsymbol{L}^{-1} = \begin{pmatrix} \frac{23}{360} & -\frac{13}{90} & \frac{53}{360} \\ \frac{19}{180} & \frac{1}{45} & -\frac{11}{180} \\ -\frac{37}{360} & \frac{17}{90} & -\frac{7}{360} \end{pmatrix}$$

$$\boldsymbol{L}^{-1}\mathbf{1} = \begin{pmatrix} \frac{1}{15} \\ \frac{1}{15} \\ \frac{1}{15} \end{pmatrix}$$

$$\mathbf{1}'\boldsymbol{L}^{-1} = \begin{pmatrix} \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \end{pmatrix}$$

	trace	det	rank	eigenvalues
L	15	360	3	[15, √24·*i*, -√24·*i*]
$\boldsymbol{L}^{-1}$	1/15	1/360	3	[1/15, 1/(-√24·*i*), 1/(√24·*i*)]

$$\operatorname{tr}\left(\boldsymbol{F}\boldsymbol{L}^{-1}\right) = \frac{1}{15}$$

$\boldsymbol{L}^{-1}$ is magic.

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Inverses II

D is a singular matrix

$$\boldsymbol{D}^{+} = \begin{pmatrix} \frac{55}{544} & -\frac{201}{2720} & -\frac{167}{2720} & \frac{173}{2720} \\ \frac{37}{2720} & -\frac{31}{2720} & -\frac{13}{544} & \frac{139}{2720} \\ -\frac{99}{2720} & \frac{21}{544} & \frac{71}{2720} & \frac{3}{2720} \\ -\frac{133}{2720} & \frac{207}{2720} & \frac{241}{2720} & -\frac{47}{544} \end{pmatrix}$$

$$\boldsymbol{D}^{+}\mathbf{1} = \begin{pmatrix} \frac{1}{34} \\ \frac{1}{34} \\ \frac{1}{34} \\ \frac{1}{34} \end{pmatrix}$$

$$\mathbf{1}'\boldsymbol{D}^{+} = \begin{pmatrix} \frac{1}{34} & \frac{1}{34} & \frac{1}{34} & \frac{1}{34} \end{pmatrix}$$

(***D***$^+$ denotes the Moore-Penrose inverse of ***D***)

	trace	det	rank	eigenvalues
D	34	0	3	[34, 0, 8, -8]
D$^+$	1/34	0	3	[1/34, 0, 1/10, -1/10]

$$\mathrm{tr}\left(\boldsymbol{F}\boldsymbol{D}^{+}\right) = \frac{1}{34}$$

D$^+$ is magic.

Inverses III

S is a nonsingular matrix

$$S^{-1} = \begin{pmatrix} -\frac{13}{99} & -\frac{2}{99} & \frac{20}{99} & -\frac{2}{99} \\ \frac{67}{495} & \frac{353}{495} & -\frac{263}{495} & -\frac{142}{495} \\ -\frac{43}{495} & -\frac{362}{495} & \frac{287}{495} & \frac{133}{495} \\ \frac{56}{495} & \frac{34}{495} & -\frac{109}{495} & \frac{34}{495} \end{pmatrix}$$

$$S^{-1}\mathbf{1} = \begin{pmatrix} \frac{1}{33} \\ \frac{1}{33} \\ \frac{1}{33} \\ \frac{1}{33} \end{pmatrix}$$

$$\mathbf{1}'S^{-1} = \begin{pmatrix} \frac{1}{33} & \frac{1}{33} & \frac{1}{33} & \frac{1}{33} \end{pmatrix}$$

	trace	det	rank	eigenvalues
S	33	-1485	4	[33, ≈0.8445, ≈-7.734, ≈6.890]
S^{-1}	203/ 165	-1/1485	4	[1/33, ≈1/0.8445, ≈-1/7.734, ≈1/6.890]

$$\mathrm{tr}\left(FS^{-1}\right) = -\frac{193}{165}$$

S^{-1} is only semi-magic!

References

- K. Schmidt and G. Trenkler, *The Moore-Penrose Inverse of a Semi-Magic Square is Semi-Magic*, **International Journal of Mathematical Education in Science and Technology 32 (2001)**, 624-629.

- G. Trenkler, K. Schmidt, and D. Trenkler, *A Simple Parameterization of 3x3 Magic Squares*, **International Journal of Mathematical Education in Science and Technology 43 (2012)**, 128-133.