Vector Spaces and Subspaces

(Institution D)

PART I: Vector Spaces

Definition: A **vector space** is a nonempty set *V* of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors **u**, **v**, and **w** in *V* and for all scalars *r* and *s*.

(1) For any v, w ∈ V, their **vector sum** is an element of V.
(2) If v, w ∈ V, u + v = v + u
(3) For any u, v, w ∈ V, u + (v + w) = (u + v) + w
(4) There is a **zero vector, 0** ∈ V, such that for all **u** ∈ *V* we have u + 0 = 0 + u = u.
(5) Each has an **additive inverse** w ∈ V such that w + v = 0.
(6) If r is a scalar, that is a member of ℝ and v ∈ V then the **scalar multiple** is in V.
(7) If v, w ∈ V then (r + s) · v = r · v + s · v
(8) If v, w ∈ V then r · (v + w) = r · v + r · w
(9) If v, w ∈ V, (rs) · v = r · (s · v)
(10) For any v ∈ V, 1 · v = v

Using this definition determine whether the following examples represent a vector space. In each case, circle your response and provide justification when necessary.

1. The set $\mathbb{R}^2$, where the operations "+" and "·" have their usual meaning:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} \text{ and } r \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} rx_1 \\ rx_2 \end{pmatrix}.$$

   Answer: YES NO Justify your answer.

2. The subset of $\mathbb{R}^3$ that is a plane through the origin $P = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} | x + y + z = 0\}$ and operations "+" and "·" have their usual meaning.

   Answer: YES NO If your answer is NO, please list the axioms that fail.

3. {0} – The set of zero vector

   Answer: YES NO Justify your answer.

4. $F(\mathbb{R})$: the set of all functions $f: \mathbb{R} \longrightarrow \mathbb{R}$.

   Answer: YES NO Justify your answer.

5. $C(\mathbb{R})$: the set of all continuous functions $f: \mathbb{R} \longrightarrow \mathbb{R}$.

   Answer: YES NO Justify your answer.

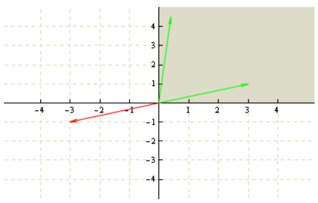
6. $\mathcal{P}$: all polynomials $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$

   Answer: YES NO Justify your answer.

7. The subset of $\mathbb{R}^3$ that is a plane parallel to the *xy*-plane, $P = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid z = 5\}$ where operations "+" and "·" have their usual meaning as described in previous example.

   Answer: YES NO If your answer is NO, please list at least one axiom that fails.

8. A set of all vectors in the first quadrant:



   Answer: YES NO If your answer is NO, please list at least one axiom that fails.

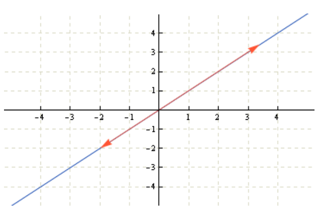
PART II: Subspaces

A *subspace* of a vector space is defined in the following way:

Definition: A subspace of a vector space *V* is a subset *H* of *V* that has three properties:

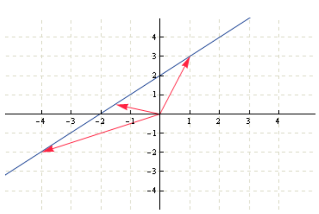
a. The zero vector of *V* is in H
b. *H* is closed under vector addition. That is, for each $\boldsymbol{u}$ and $\boldsymbol{v}$ in *H*, the sum $\boldsymbol{u}+\boldsymbol{v}$ is in *H*.
c. *H* is closed under multiplication by scalar. That is, for each $\boldsymbol{u}$ in *H* and each scalar c, the vector $c\boldsymbol{u}$ is in *H*.

1. Is the line given in the picture a subspace of $\mathbb{R}^2$?
   (a)



Answer: YES NO Justify your answer.

(b)



Answer: YES NO Justify your answer.

2. Go back and examine all of the examples given in PART I (for vector spaces). Can you identify any two sets that represent a vector space and its subspace?

Answer:

3. Give at least two more examples of a vector space and its subspace. If you wish, you may use the examples given in PART I.