Linear Maps

(Institution B)

Make sure that you explain all your answers. Your solutions must be written up **clearly, legibly, in complete sentences,** primarily focusing on **explaining your reasoning**. For this particular assignment, most credit will be awarded for satisfying these conditions. As usual, F will denote either $\mathbb{R}$ or $\mathbb{C}$.

1. a. Give an example of a linear map $T: R^2 \to R^2$ such that $T(2,3) = (7,8)$ and $T(3,4) = (10,11)$. (If such a linear map does not exist, explain why.) For example, does there exist $(a, b) \in R^2$ such that $T(a, b) = (1,1)$?

   b. How many linear maps $T: R^2 \to R^2$ with $T(2,3) = (7,8)$ and $T(3,4) = (10,11)$ exist?

2. a. Give an example of a linear map $T: R^3 \to R^2$ such that $T(2,3,1) = (7,8)$ and $T(3,4,2) = (10,11)$ (If such a linear map does not exist, explain why.)

   b. How many linear maps $T: R^3 \to R^2$ such that $T(2,3,1) = (7,8)$ and $T(3,4,2) = (10,11)$ exist?

3. Let U, V be finite dimensional vector spaces over F and let $u_1, \dots, u_n$ be a list of linearly independent vectors in U and $v_1, \cdots v_n \in V$. Prove that there exists a linear map $T: U \to V$ such that $T(u_i) = v_i$ for all $i = 1, \dots, n$.
4. Let U, V be finite dimensional vector spaces over F and let $u_1, \dots, u_n$ be a list of vectors in U. Assume that for every $v_1, \cdots v_n \in V$ there exists a linear map $T: U \to V$ such that $T(u_i) = v_i$ for all $i = 1, \dots, n$. Prove that the vectors $u_1, \dots, u_n$ are linearly independent.
5. Let U, V be finite dimensional vector spaces over F, W a subspace of U, and let $T: W \to V$ be a linear map. Prove that there exists a linear map $S: U \to V$ that extends T. (S extends T means $S(w) = T(w)$ for all w.)