Contributed Paper Session (11-11:20 am Thursday Jan 5)

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- Project context
- NDSU Module 2: *Linear Maps*
- Action Process Object Schema (APOS) theoretical framework

(More information about this project will be given at the NSF Poster Session 2-4 pm Thursday afternoon in Veteran's Auditorium, 2nd Floor, Hynes Convention Center)

Linear algebra In New Environments (LINE) is a project funded by the NSF CCLI program (NSF DUE-#0837050) at four universities: Brooklyn College—CUNY, Georgia State University, Grandview University, and North Dakota State University. The goal of the project is to make use of research on teaching and learning and a specific theoretical framework, APOS, in undergraduate linear algebra courses. The mechanism we use is the development of modules that involve core ideas found in undergraduate linear algebra courses and reflect educational theory about the teaching and learning of collegiate mathematics.

The handout for this session has one of the project modules developed and used at NDSU and a sketch of the APOS research framework. Please take a few minutes to scan the material in that handout.

This module was developed by the course instructor, a research mathematician at NDSU who taught the linear algebra course, working with the LINE project researchers. The instructor chose the topic, *linear maps*, as one of the central themes of his course and proposed a series of problems that students would work on at the start of the study of that material. The textbook used in this course was Sheldon Axler's *Linear Algebra Done Right* (2nd Edition, 1997 corrected printing 2004). The instructor worked with the project researchers to refine his problems to a set that would be

- Accessible to students in his class prior to formal instruction in this component of the course
- Aligned with the APOS framework for concept development in mathematics
- Useful to assess student understanding of the material as a guide to instruction

The instructor and researchers read and discussed a series of mathematics education research manuscripts during the summer prior to the course to provide the context for this work.

Look at the module to see how it reflects the APOS framework of the hierarchical development of conceptual understanding of specific content (this framework has been applied to many important collegiate mathematics concepts).

Problem 1a primarily require understanding at an *action* level. Students are given specific vectors and use prior knowledge of functions and systems of equations to compute an appropriate map.

Problem 1b requires both *action* and *process* conceptions because the student must be able to imagine how any other linear transformation would map the given vectors while recognizing from their computations in part (a) that the given vectors determine the map.

Problems 2a and 2b again have students draw on *action* and *process* conceptions in a setting that is less determined by the provided information. Again, students will need to use knowledge from prior work with systems of equations to solve the problems. Parts 1b and 2b provide a sort of transition to the purely process Problem 3 by referring to a specific action that the students have carried out.

Problem 3 requires a process conception that could be based on reflection of Problems 1 and 2.

Problem 4 also uses a process conception.

The proof in **Problem 5** draws on an object conception of linear map that must be de encapsulated to analyze the processes that created the cognitive object.

Problem Set 2 (Linear Maps)

Make sure that you explain all your answers. Your solutions must be written up **clearly, legibly, in complete sentences**, primarily focusing on **explaining your reasoning.** For this particular assignment, most credit will be awarded for satisfying these conditions. As usual, $\mathbb{F}$ will denote either $\mathbb{R}$ or $\mathbb{C}$.

1. (a) Give an example of a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(2,3) = (7,8)$ and $T(3,4) = (10,11)$. (If such a linear map does not exist, explain why.) For your example, does there exist $(a,b) \in \mathbb{R}^2$ such that $T(a,b) = (1,1)$?

   (b) How many linear maps $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(2,3) = (7,8)$ and $T(3,4) = (10,11)$ exist?

2. (a) Give an example of a linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(2,3,1) = (7,8)$ and $T(3,4,2) = (10,11)$. (If such a linear map does not exist, explain why.)

   (b) How many linear maps $T : \mathbb{R}^3 \to \mathbb{R}^2$ with $T(2,3,1) = (7,8)$ and $T(3,4,2) = (10,11)$ exist?

3. Let U, V be finite dimensional vector spaces over $\mathbb{F}$ and let $(u_1, \ldots, u_n)$ be a list of linearly independent vectors in U and $v_1, \ldots, v_n \in V$. Prove that there exists a linear map $T : U \to V$ such that $T(u_i) = v_i$ for all $i = 1, \ldots, n$.

4. Let U, V be finite dimensional vector spaces over $\mathbb{F}$ and let $(u_1, \ldots, u_n)$ be a list of vectors in U. Assume that for every $v_1, \ldots, v_n \in V$ there exists a linear map $T : U \to V$ such that $T(u_i) = v_i$ for all $i = 1, \ldots, n$. Prove that the vectors $u_1, \ldots, u_n$ are linearly independent.

5. Let U, V be finite dimensional vector spaces over $\mathbb{F}$, W a subspace of U, and let $T : W \to V$ be a linear map. Prove that there exists a linear map $S : U \to V$ that extends T. (S extends T means $S(w) = T(w)$ for all $w \in W$.)

Action Process Object Schema Theory (APOS)

Excerpted from *Integrating Learning Theories and Application-based Modules in Teaching Linear Algebra* (Martin, Loch, Cooley, Dexter, Vidakovic; 2010)

APOS theory is a theoretical perspective of learning based on an interpretation of Piaget's constructivism and poses descriptions of mental constructions that may occur in understanding a mathematical concept. These constructions are called actions, processes, objects, and schema.

An **action** is a transformation of a mathematical object according to an explicit algorithm seen as externally driven. It may be a manipulation of objects or acting upon a memorized fact.

When one reflects upon an action, constructing an internal operation for a transformation, the action begins to be interiorized. A **process** is this internal transformation of an object. Each step may be described or reflected upon without actually performing it. Processes may be transformed through reversal or coordination with other processes.

There are two ways in which an individual may construct an **object**. A person may reflect on actions applied to a particular process and become aware of the process as a totality. One realizes that transformations (whether actions or processes) can act on the process, and is able to actually construct such transformations. At this point, the individual has reconstructed a process as a cognitive object. In this case we say that the process has been *encapsulated* into an object. One may also construct a cognitive object by reflecting on a *schema*, becoming aware of it as a totality. Thus, he or she is able to perform actions on it and we say the individual has *thematized* the schema into an object. With an object conception one is able to de-encapsulate that object back into the process from which it came, or, in the case of a thematized schema, unpack it into its various components. Piaget & Garcia [13] indicate that thematization has occurred when there is a change from usage or implicit application to consequent use and conceptualization.

A **schema** is a collection of actions, processes, objects, and other previously constructed schemata which are coordinated and synthesized to form mathematical structures utilized in problem situations. Objects may be transformed by higher-level actions, leading to new processes, objects, and schemata. Hence, reconstruction continues in evolving schemata.

To illustrate different conceptions of the APOS theory, imagine the following 'teaching' scenario. We give students multi-part activities in a technology supported environment. In particular, we assume students are using Maple in the computer lab. The multi-part activities, focusing on vectors and operations, in Maple begin with a given Maple code and drawing. In case of scalar multiplication of the vector, students are asked to substitute one parameter in the Maple code, execute the code and observe what has happened. They are asked to repeat this activity with a different value of the parameter. Then students are asked to predict what will happen in a more general case and to explain their reasoning. Similarly, students may explore addition and subtraction of vectors. In the next part of activity students might be asked to investigate about the commutative property of vector addition.

Based on APOS theory, in the first part of the activity—in which students are asked to perform certain operation and make observations—our intention is to induce each student's *action conception* of that concept. By asking students to imagine what will happen if they make a certain change–but do not physically perform that change–we are hoping to induce a somewhat higher level of students' thinking, the *process level*. In order to predict what will happen students would have to imagine performing the action based on the actions they performed before (reflective abstraction). Activities designed to explore on vector addition properties require students to encapsulate the process of addition of two vectors into an object on which some other action could be performed. For example, in order for a student to conclude that $u + v = v + u$, he/she must encapsulate a process of adding two vectors $u + v$ into an object (resulting vector) which can further be compared [action] with another vector representing the addition of $v + u$.

As with all theories of learning, APOS has a limitation that researchers may only observe externally what one produces and discusses. While schemata are viewed as dynamic, the task is to attempt to take a snap shot of understanding at a point in time using a genetic decomposition. A genetic decomposition is a description by the researchers of specific mental constructions one may make in understanding a mathematical concept. As with most theories (economics, physics) that have restrictions, it can still be very useful in describing what is observed.

Further Information

We asked the instructor how he thought our readings and discussions of learning theories in relation to linear algebra had influenced his thinking about best teaching practices. He responded

> *I have gained some awareness when crafting mathematical questions to my students. I have always thought that going from simple to complex is a good approach. After these readings I consciously divide this evolution from simple to complex into several layers that require increasing levels of understanding.*
>
> *Learning is a personal activity (and so is teaching). Different individuals have different ways to learn mathematics. This often results in different ways of understanding the same mathematical concept. I do not mean different levels of understanding, just different ways to look at a concept. Each individual using his/her own "hardware" (self-explanatory) uses personal "software" (logical thinking) to process knowledge (learning process) delivered through the activity of teaching (self-teaching included). As a result, one relates this knowledge to previous knowledge, connects and relates new concepts and facts with previously processed concepts and facts. There are different ways to make these connections; the mathematical proofs are the connections of the highest quality. Teaching does not deliver thinking (which is a process that takes place in the student's "hardware"), but very often, by processing the knowledge delivered (i.e. learning), the "software" improves itself. This "byproduct" of teaching (the self-adjustment and improvement of the "software"), on the long run, is by far the most important output of the learning process. The reverse process is also possible: the "software" can self-downgrade when not used frequently. I see learning as a process deeply integrated with (and therefore greatly influenced by) the "hardware" and "software" components.*

Other Modules

- Systems of Equations
- Vector Spaces and Subspaces
- Linear Independence, Span
- Basis, Change of Basis, Dimension
- Inner Products
- Operators on Complex Vector Spaces

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