Group Theory in Linear Algebra

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Math 436 (Linear Algebra) at Penn State Erie

Problems:

1. three credit linear algebra course;
2. **most** students have taken two-credit introduction (Math 220);
3. students from a variety of majors;
4. students take senior-level courses in random order.

Partial solution:

Maple labs which introduce elementary linear algebra topics together with unifying concepts.

Cayley's theorem: the usual undergraduate presentation.

Every group with n elements is isomorphic to a group of $n \times n$ permutation matrices.

The regular representation of the cyclic group with six elements:

$$\mathbb{Z}_6 \quad \cong \quad \langle x : x^6 = e \rangle$$

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Cayley's theorem (a liberal presentation)

Suppose G is a group of order n. G is isomorphic to a group of $m \times m$ permutation matrices, for at least one $m \leq n$.

A faithful blocking of the dicyclic group with order 12:

$$\mathbb{Z}_3 \rtimes \mathbb{Z}_4 \cong \langle a, b : a^3 = b^4; bab^{-1} = a^{-1} \rangle.$$

$$A = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{bmatrix} \quad ; \quad B = \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{bmatrix}$$

Advantages of blocked representations of groups

1. connects **abstract algebra** to **linear algebra**;
2. encourages visual treatment of matrix actions on vectors;
3. treats group elements as **objects** to manipulate;
4. allows easy manipulation of group elements (Maple);
5. allows visual treatment of homomorphisms;
6. treats subgroups and normal subgroups as **objects**;
7. allows visual classification of small groups.

A Maple lab experience in linear algebra (Math 436)

$$A = \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} \quad ; \quad B = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{bmatrix}$$

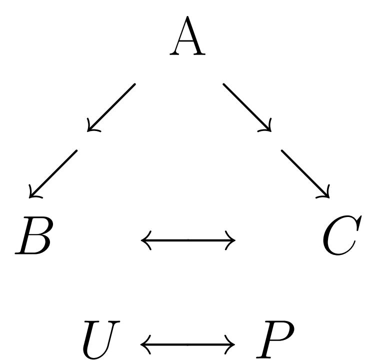
1. determine the number of multiples of A and B;
2. verify that multiplication is associative (in this set);
3. explain **why** the set is closed under multiplication;
4. show that the set contains a multiplicative identity;
5. show that every element of the set has a unique inverse;
6. show that multiplication in this set is not commutative;
7. summarize calculations as a definition (group) – identify the group above.

Student responses:

1. helped to understand multiplication in unfamiliar context;
2. helped to understand inverses as more than formula;
3. students explained multiplication to each other;
4. some students started explaining closure and inverses using block structure;
5. some made conjectures about the order of elements based on the block structure;
6. students who had completed abstract algebra recognized the group, and asked how the matrices could represent action on a triangle.

Follow-up:

1. action on a vector. $$\begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \\ U \\ P \end{bmatrix} = \begin{bmatrix} A \\ C \\ B \\ P \\ U \end{bmatrix}$$



2. conjugation (similarity);

3. eventually, canonical forms.

More about faithful blockings

Becker, "Do Normal Subgroups Have Straight Tails?" American Math Monthly, Volume 112, No. 5, pp 440-449.

If G is a semidirect product of H by K, it admits a faithful blocking in $|H| \times |H|$ and $|K| \times |K|$ blocks.

Results by Mark Medwid:

- If a group, G, is a direct product $G = H_1 \times H_2 \times \cdots \times H_k$, then G admits a faithful blocking with k blocks.
- $S_n, (n > 2)$, can be written as a set product of two non-normal subgroups. It admits a faithful blocking generated by cosets of those subgroups.
- Faithful blockings can simplify explanations of these concepts:
  the purpose of groups; normal subgroups; homomorphisms and isomorphisms; images and kernels; classification of small groups; canonical forms in linear algebra; symmetry groups in geometry; etc.

Construction of faithful blockings

Suppose G is a semi-direct product of two subgroups, $G = N \rtimes H$.

Define a vector of left cosets for each subgroup:
$\widehat{G/H} = [eH, (n_1)H, (n_2)H, \ldots]^T$;
$\widehat{G/N} = [eN, (h_1)N, (h_2)N, \ldots]^T$.

For each $z \in G$, let $\Phi_H(z)$ be the permutation matrix taking $\widehat{G/H}$ to $[zH, (zn_1)H, (zn_2)H \ldots]^T$; define $\Phi_N(z)$ similarly.

The resulting mapping is a faithful representation of the group:

$$z \longrightarrow \begin{bmatrix} \Phi_N[z] & 0 \\ 0 & \Phi_H[z] \end{bmatrix}$$

In our example, the subgroup $\langle a \rangle$ is normal, with 4 cosets; $\langle b \rangle$ is abnormal, with 3.

$$A = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{bmatrix} \quad ; \quad B = \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{bmatrix}$$

Computer lab: Homomorphisms in Maple

The dicyclic group admits an "obvious" homomorphisms.

$$f : \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$f : \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Kernel of homomorphism is a normal subgroup: $Kernel(f) = \langle A \rangle \cong \mathbb{Z}_3$.

$Image(f) \cong \langle B \rangle \cong \mathbb{Z}_4$.

Computer lab: Homomorphisms in Maple

The dicyclic group admits a not-so-obvious homomorphisms.

$$g : \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$g : \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Kernel of homomorphism is a normal subgroup: $Kernel(g) = \{I, B^2\} \cong \mathbb{Z}_2$.

$Image(g) \cong S_3$ **is generated by the block of** A **and the tail of** B**.**

The Behrend College Mathematics Universe

A few recent courses:

- Math 220: Matrices
- Math 311: Concepts of Discrete Mathematics
- Math 427: Geometry
- Math 428: Geometry for Teachers
- Math 435: Abstract Algebra
- Math 436: Linear Algebra

Common topics:

- Binary operation
- identity and inverse
- group action (matrix action)
- similarity (conjugation)
- basis (generating set)
- homomorphism
- permutation group