Enhancing Students' Attitudes towards Linear Algebra with Technology

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When teaching linear algebra we have to deal with the following well-known problem: while the level of mathematical skills required to work with examples is generally low (students only need to add, subtract and multiply), the number of calculations is usually large. Therefore, working with examples is time-consuming and error-prone if done by hand. Students get tired quickly and lose interest in this increasingly important area of mathematics. The Schmalkalden University Faculty of Business and Economics therefore decided to move its introductory linear algebra course from the classroom to the PC lab, and purchased a Computer Algebra System license that also allows its use on the students' own PCs. A collection of functions was then developed to facilitate teaching by providing functions for the computation of zero matrices and vectors, matrices and vectors of ones, identity matrices, as well as idempotent and orthogonal matrices, "just-in-time", whenever they are needed during the course. Examples will show how beneficial it is, from the very basics to the more advanced topics, to sit in front of a PC in an introductory linear algebra course.

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1 Introduction

For many years the introductory linear algebra course at the Schmalkalden University Faculty of Business and Economics was taught in a regular classroom equipped with blackboard and chalk as well as an overhead projector.

In 2000 the faculty began equipping classrooms with beamers which could be connected to the notebook of a teacher who was then able to project the computer screen onto a silver screen.

A little later the faculty purchased a license for the Computer Algebra System *DERIVE* that covered all PCs in the faculty's two PC labs as well as the private PCs of the students.

Since 2007 the course has been held in the PC lab only. This limits the number of students per group to 40 (sitting in front of 20 PCs) and therefore comes at a price (three groups instead of two).

Since then the final exams have been held in the PC labs (which again comes at a price: exams now have to take place in many small groups since naturally only one student can sit in front of a PC during an exam). They contain some questions which can only be solved using a PC, but also questions which take longer to solve with a PC than without. All questions still have to be answered on the exam papers alone; nothing has to be stored electronically.

The *DERIVE* function collection used in the following demonstrations can be downloaded from

http://www.fh-schmalkalden.de/schmalkaldenmedia/emad.mth

2 Working with Zero Vectors and Zero Matrices

O(m,n) generates an $m \times n$ zero matrix

o(n)   generates an $n \times 1$ zero vector

$$\underset{m \times n}{A} - \underset{m \times n}{A} = \underset{}{O} \qquad \underset{m \times n}{A} + \underset{m \times n}{O} = A$$

$$\underset{m \times n}{A} \underset{n \times l}{O} = \underset{m \times l}{O} \qquad \underset{l \times m}{O} \underset{m \times n}{A} = \underset{l \times n}{O}$$

$$\underset{m \times n}{O} = \underset{m \times 1}{o} \underset{1 \times n}{o'} \qquad \underset{m \times n}{O} \underset{n \times p}{O} = \underset{m \times p}{O}$$

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#1:    LOAD(C:\Program Files\TI Education\Derive 6\Math\EMAD.mth)

              ⎡ 1  2  3 ⎤
#2:    A :=   ⎢ 4  5  6 ⎥
              ⎣ 7  8  9 ⎦
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$$\text{\#3:} \qquad A - A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{\#4:} \qquad O(2,3) \cdot A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{\#5:} \qquad A \cdot o(3) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

3 Working with Matrices and Vectors of Ones

$J(m,n)$ generates an $m \times n$ matrix of ones

$1(n)$ generates an $n \times 1$ vector of ones

$$\underset{1 \times n}{a'} \underset{n \times 1}{1} = 1'a = \sum_{i=1}^{n} a_i \qquad \underset{1 \times m}{1'} \underset{m \times n}{A} \underset{n \times 1}{1} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$$

$$\underset{m \times n}{A} \underset{n \times 1}{1} = \begin{pmatrix} \sum_{j=1}^{n} a_{1j} \\ \vdots \\ \sum_{j=1}^{n} a_{mj} \end{pmatrix} \qquad \underset{1 \times m}{1'} \underset{m \times n}{A} = \begin{pmatrix} \sum_{i=1}^{m} a_{i1} & \cdots & \sum_{i=1}^{m} a_{in} \end{pmatrix}$$

$$\underset{m \times n}{J} = \underset{m \times 1}{1} \underset{1 \times n}{1'} \qquad \underset{m \times n}{J} \underset{n \times p}{J} = n \underset{m \times p}{J}$$

#6: $a := \begin{bmatrix} a1 \\ a2 \end{bmatrix}$

#7: $a' \cdot 1(2) = [[a1 + a2]]$

#8: $1(3)' \cdot A \cdot 1(3) = [[45]]$

#9: $A \cdot 1(3) = \begin{bmatrix} 6 \\ 15 \\ 24 \end{bmatrix}$

#10: $1(3)' \cdot A = [[12, 15, 18]]$

#11: $J(2, 3) \cdot J(3, 4) = \begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \end{bmatrix}$

4   Working with Identity Matrices

I(n)   generates an $n \times n$ identity matrix

$$\underset{m \times n}{A} \; \underset{n \times n}{I} = \underset{m \times m}{I} \; \underset{m \times n}{A} = A$$

#12:
$$I(3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

#13:
$$A \cdot I(3) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

#14:
$$I(2) \cdot a = \begin{bmatrix} a1 \\ a2 \end{bmatrix}$$

#15:
$$I(2) \cdot I(2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

5 Working with Idempotent Matrices

Any square matrix A with the property $AA = A$ is an idempotent matrix.

IDEM(n) generates an idempotent $n \times n$ matrix

The formula used in this function:

$$\underset{n \times n}{I} - \frac{1}{n} \underset{n \times n}{J}$$

#16:
$$\text{IDEM(2)} = \begin{bmatrix} \dfrac{1}{2} & -\dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{1}{2} \end{bmatrix}$$

#17:
$$\text{IDEM(3)} = \begin{bmatrix} \dfrac{2}{3} & -\dfrac{1}{3} & -\dfrac{1}{3} \\ -\dfrac{1}{3} & \dfrac{2}{3} & -\dfrac{1}{3} \\ -\dfrac{1}{3} & -\dfrac{1}{3} & \dfrac{2}{3} \end{bmatrix}$$

#18:
$$\text{IDEM(3)} - \text{IDEM(3)} \cdot \text{IDEM(3)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The following property holds for the Moore-Penrose inverse $\underset{n \times m}{A^+}$ of

any matrix $\underset{m \times n}{A}$:

A^+A and AA^+ (as well as $I - A^+A$ and $I - AA^+$) are idempotent

matrices.

MPI(A) computes Moore-Penrose inverse of any $m \times n$ matrix A

#19:   LOAD(C:\Program Files\TI Education\Derive 6\Math\MP.mth)

$$\#20: \quad \text{MPI(A)} = \begin{bmatrix} -\dfrac{23}{36} & -\dfrac{1}{6} & \dfrac{11}{36} \\[2mm] -\dfrac{1}{18} & 0 & \dfrac{1}{18} \\[2mm] \dfrac{19}{36} & \dfrac{1}{6} & -\dfrac{7}{36} \end{bmatrix}$$

$$\#21: \quad \text{MPI(A)} \cdot \text{A} = \begin{bmatrix} \dfrac{5}{6} & \dfrac{1}{3} & -\dfrac{1}{6} \\[2mm] \dfrac{1}{3} & \dfrac{1}{3} & \dfrac{1}{3} \\[2mm] -\dfrac{1}{6} & \dfrac{1}{3} & \dfrac{5}{6} \end{bmatrix}$$

#22:   RANK(MPI(A)·A) = 2

#23:   TRACE(MPI(A)·A) = 2

6 Working with Orthogonal Matrices

Any square matrix A with the property $A' = A^{-1}$ is an orthogonal matrix.

ORTH(a) generates an orthogonal $n \times n$ matrix from any $n \times 1$ vector $a \neq o$

The formula used in this function:

$$\underset{n \times n}{I} - 2 \underset{n \times 1}{a} \underset{1 \times n}{a'}$$

generates an orthogonal matrix if $a'a = 1$. Hence, the vector a which is passed as parameter will be transformed within the ORTH-function such that it is of length 1.

#24: b := $\begin{bmatrix} \dfrac{3}{5} \\ \dfrac{4}{5} \end{bmatrix}$

#25:   b'·b = [[1]]

#26:   ORTH(b) = $\begin{bmatrix} \dfrac{7}{25} & -\dfrac{24}{25} \\ -\dfrac{24}{25} & -\dfrac{7}{25} \end{bmatrix}$

For any orthogonal matrix we have

$$A'A = A^{-1}A = I \,,$$

#27: $$ORTH(b)^{-1} - ORTH(b)' = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

#28: $$ORTH(b)' \cdot ORTH(b) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Another example:

#29: $$ORTH(1(2)) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

#30: $$ORTH(1(2))^{-1} - ORTH(1(2))' = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

#31: $$ORTH(1(2))' \cdot ORTH(1(2)) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Any column of an orthogonal matrix has length 1, and any two are pairwise orthogonal. Since both examples are two-dimensional, its column vectors are lying on the unit circle and form a right angle. The graph also shows the vector **b** from the first example.
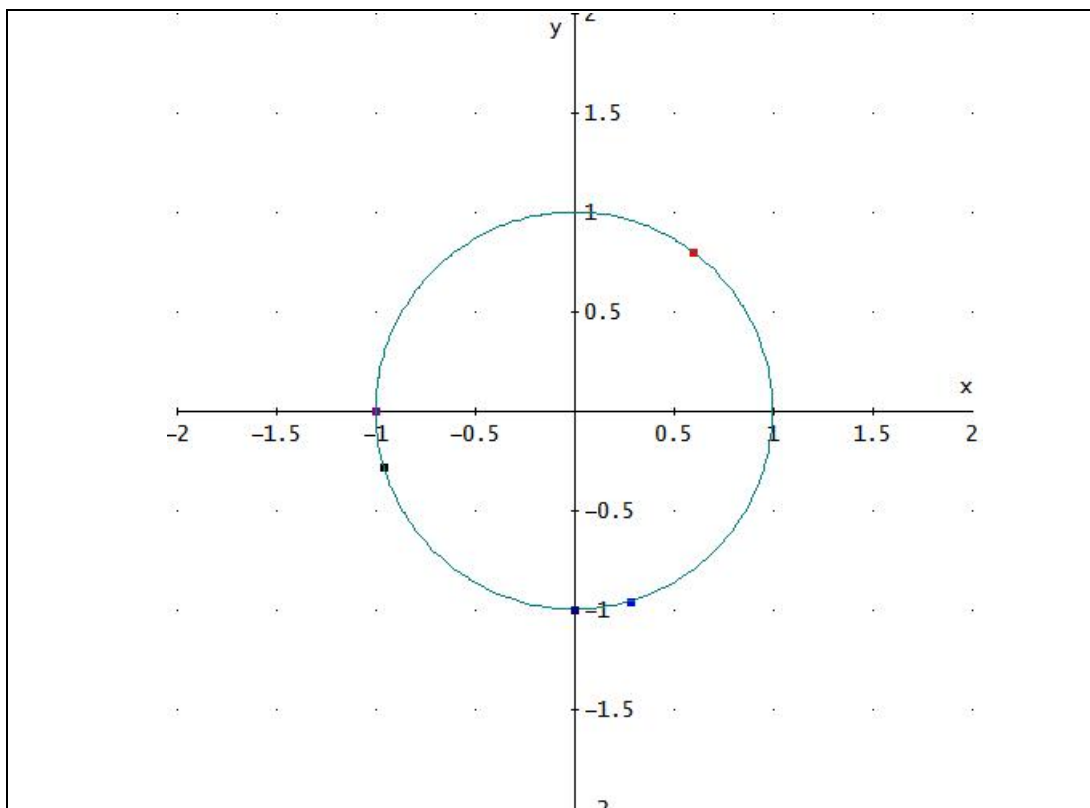
Computation of Moore-Penrose Inverse

Schmidt, K., 2008, Computing the Moore-Penrose Inverse of a Matrix with a Computer Algebra System. *International Journal of Mathematical Education in Science and Technology*, **39**, 557–562.