Introducing Linear Algebra to Middle School Students

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Abstract: A six week course of half hour sessions before school was used to introduce the ideas of linear algebra to middle school students. The course developed concepts in the context of two concrete examples: digital animation and dynamic population models. The course was not intended to be comprehensive, but rather to give students a taste of the usefulness of linear algebra for the conceptualization as well as implementation of solutions to problems in contemporary applications. It is conjectured that familiarity with concrete situations such as these will increase interest and motivation for subsequent courses in linear algebra.

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Matrix Math Course I Overview

Prerequisites: basic arithmetic, basic geometry (angles, circle, rotations)

Course objectives:

Each day of the course has 2 or 3 very specific goals. By the end of the course, students will understand what matrices are and why they are useful. They will have learned how to do basic arithmetic with matrices, and they will have used matrix multiplication to rotate a figure in the plane – as an example of digital animation.

Course Materials:

1. A syllabus listing the topic and the specific goals for each of 12 lessons with 2 days for review and assessment.
2. Activity sheets for each lesson.
3. A course summary sheets for review/assessment.

A Matrix Math Short Course II covered the topic of dynamic systems, with repeated multiplication to describe changes evolving over time, with a focus on digital animation and population models. Materials for this second three week course are not included here.

Matrix Math Short Course I Syllabus

As organized and taught by Pamela Coxson

Day 1

Goals:
1. Explain what is meant by the mathematical term "matrix".
2. Give the dimensions of a matrix in the proper order.
3. List several fields of work or study in which matrices are important.

Day 2

Topic: The Dot Product (a special case of matrix multiplication)

Goals:
1. Be able to calculate the dot product of a row vector and a column vector.
2. Learn how to recognize vectors that cannot be multiplied in this way.
3. Give at least 2 examples of dot products in everyday life.

Days 3-4

Topic: Matrix Multiplication

Goals:
1. Know how the elements of a matrix product were calculated.
2. Know how to multiply the rows of one matrix times the columns of a 2nd matrix to obtain the product.
3. Know that matrix multiplication is not commutative.

Day 5

Topic: Matrix Transformations in a Coordinate Plane

Goals:
1. Know how to place the x and y coordinates of a point into a column vector.
2. Multiply a 2x2 matrix times the vectors of all the corners (vertices) of a geometric shape and plot the resulting points to see the result of this transformation.
3. Know that the vectors of the corners can all be put into one matrix for faster calculation.

Day 6

Topic: Special Matrices

Goals:
1. Know how to find the identity matrix (the "1" of matrix multiplication).
2. Learn about the Zero matrix and other nilpotent matrices.
3. Learn how to identify diagonal, symmetric and skew symmetric matrices.

Day 7

Topic: Rotation of the points (1,0) and (0,1)

Goals:
1. Rotate the point (1,0) 30 degrees and find its new coordinates.
2. Rotate the point (0,1) 30 degrees and find its new coordinates.

Day 8

Topic: Matrix Transformation for the 30 degree Rotation

Goals:
1. Find a matrix transformation that moves (1,0) and (0,1) to the rotated positions.
2. Use this transformation to rotate a figure of your choice.

Day 9-10

Topic: Rotations of Any Degree – Sine and Cosine

Goals:
1. Find the coordinates (x=cosine; y=sine) for rotating (1,0) multiples of 15°.
2. Know how to form the transformation matrix for any rotation.

Day 11

Topic: Rotate a figure by 75°.

Goals:
1. Find the coordinates of the figure.
2. Find the matrix for rotation by 75°.
3. Use this transformation matrix to rotate the figure in the xy plane.

Day 12

Topic: Draw your own figure and rotate it 75°.

Goals:
1. Recall the matrix for rotation by 75°.
2. Use this transformation matrix to rotate your figure drawn in a coordinate plane.

Day 13-14

Topic: Review course material.

Goals:
1. Review the key concepts and skills from the course.
2. Answer practice questions.

MATRIX MATH Day 1

Goals:
1. Explain what is meant by the mathematical term "matrix".
2. Give the dimensions of a matrix in the proper order.
3. List several fields of work or study in which matrices are important.

Background for the teacher

Matrices are rectangular arrays of numbers – basically, tables of numbers. Instead of row and column headings, they are enclosed in table-sized parentheses. The activity gives examples of two tables: one with numbers representing hours of work performed by several students and a second table with costs of grocery items at two different stores. The corresponding matrices are just the numbers without the headings. Parentheses define the boundaries of the matrix.

The first question that students might ask is "why define this new object – the matrix? Why not just keep the numbers in the table?" The answer is that we plan to do some arithmetic with the matrices that will help us solve some problems and the table with all its words would be too cumbersome to work with. We will add matrices, multiply them, and even raise them to powers. We can give them letter names like A, B, and C and develop a whole new arithmetic of matrices: A+B, A*C, C^3. And the results of our arithmetic will answer questions about the situation that the numbers came from.

The first important property that we need to understand about matrices is that they have dimensions – the number of rows and the number of columns. There is a convention that the number of rows is always stated first. The following is an example of a 2 row by 3 column matrix (we say "2 by 3 matrix" and we write "2x3 matrix"):

$$\begin{pmatrix} 2 & 7 & -1 \\ 4 & 0 & 1 \end{pmatrix}$$

Giving the dimensions of a matrix is a way to describe the size of the matrix.

The next good question has to do with usefulness. When would using a matrix help us solve problems that we could not solve just as easily without matrices? Here are some of the fields of study where matrices are used: engineering, image processing, linear programming, statistics, tomography (medical imaging), and digital animation. In this course, we will help students see how matrices would have a fundamental role in digital animation.

Suggestion: Each day I gave students a sheet of paper with the goals of the day printed at the top (basically this page, without the teacher background). Then they could take notes on the page and keep the sheet in their notebooks for future reference.

MATRIX MATH Day 1 Activity

WHAT IS A MATRIX?

1) Three students are working after school selling fresh fruit snacks. They are paid $7 per hour and their hours are recorded as follows:

	Mon.	Tues.	Wed.	Thurs.	Fri.
Shawna	1 hr	1.5 hr	0.5 hr	1hr	0 hr
Joseph	1.5 hr	1.5 hr	0 hr	1.5 hr	1.5 hr
Jessica	1 hr	1.5 hr	1.5 hr	2 hr	0 hr

We can form a matrix of their daily work hours:

$$M = \begin{pmatrix} 1 & 1.5 & 0.5 & 1 & 0 \\ 1.5 & 1.5 & 0 & 1.5 & 1.5 \\ 1 & 1.5 & 1.5 & 2 & 0 \end{pmatrix}$$

This matrix has _____ rows and _____ columns. It is a ______ x ______ matrix.
These are the "dimensions" of the matrix.

2) We need fruit for snacks after the track practice. Here are the prices at a couple of local stores

	Cala.	Food Giant
Apples	$1.00	$1.10
Oranges	$1.50	$1.15
Blueberries	$2.50	$3.00
Bananas	$0.60	$0.75

Form a matrix of the prices:

$$M = \begin{pmatrix} \underline{\qquad\quad} & \underline{\qquad\quad} \\ \underline{\qquad\quad} & \underline{\qquad\quad} \\ \underline{\qquad\quad} & \underline{\qquad\quad} \\ \underline{\qquad\quad} & \underline{\qquad\quad} \end{pmatrix}$$

This matrix has _____ rows and _____ columns.

It's dimensions are ______ x ______ .

MATRIX MATH Day 2

Topic: The Dot Product (a special case of matrix multiplication)

Goals:
1. Be able to calculate the dot product of a row vector and a column vector.
2. Learn how to recognize vectors that cannot be multiplied in this way.
3. Give at least 2 examples of dot products in everyday life.

Background for the teacher

The purpose of this lesson teach the underlying mechanics of matrix multiplication (multiplying pairs of numbers and summing the products), by illustrating how it arises out of our everyday experience. This basic operation is called the dot product. Two matrices are multiplied by finding the dot product of each row (also called a row vector) of the matrix on the left with each column (column vector) of the matrix on the right.

The dot product could be called grocery store math. When you go to the store, you purchase apples at one price, bananas at another price and oranges at their price. To compute the total cost, you multiply the number of each item times the price per item (or the number of pounds times the price per pound) and then add it all up. This operation is repeated in many other areas of life as well. If we put our savings into different places (a savings account, a certificate or deposit, a money market fund), each account has its own interest rate. The interest earned is the amount in the account times the interest rate. The total interest from all accounts is the sum of these products – another dot product. The important point is that matrix multiplication was not invented by mathematicians, it is something we do all the time and it has many practical applications. Mathematicians just gave names to the parts: the vectors, the dot product, the matrix, so we could more readily see that all of these applications – grocery store sums, interest calculations and many others – are using the same basic math.

Not all vectors can be multiplied. The constraint is obvious to students after you have done a few examples: the number of numbers (we sometimes call them the "elements") in the row vector has to be the same as the number of elements in the column vector – otherwise the numbers cannot be multiplied in pairs. Every type of fruit has to have a price and every price has to go with one of the fruits. Ask students for examples of a row and column that can be multiplied and of a row and column that cannot be multiplied to check for understanding.

MATRIX MATH Day 2 Activity

Dot Product Practice Problems

1. Shopping for school supplies. Aloma buys the following supplies for her math class:
    a) 10 no. 2 pencils @ 35¢ each
    b) 5 colored dice @ 25¢ each
    c) 3 pads of lined paper @ \$1.50 each
    d) 2 packages of graph paper @ \$2 each

    Put the number of each item in a row vector and the corresponding price in the column vector below. Calculate the dot product to find the total price of these items.

$$\begin{pmatrix} \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \end{pmatrix} \begin{pmatrix} \_\_\_ \\ \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{pmatrix} =$$

2. Packages with different quantities. Norman brought out 10 packages of protractors to distribute to his classmates. The number of protractors was not the same in all packages:
    a) 1 package had 6 protractors
    b) 3 packages had 4 protractors each
    c) 7 packages had 2 protractors each

    Put the number of packages in a row vector and the corresponding number of protractors in each package in the column vector. Calculate the dot product to find the total number of protractors.

$$\begin{pmatrix} \_\_\_ & \_\_\_ & \_\_\_ \end{pmatrix} \begin{pmatrix} \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{pmatrix} =$$

3. Money in the bank. N'Tanya has put her \$1,000 savings into 3 bank accounts:
    a) a \$200 certificate of deposit earning 6% interest per year
    b) a \$500 regular savings account earning 2% interest per year
    c) a \$300 money market account earning 4% interest per year

    Put the dollar amounts into a row vector and the corresponding interest rates into a column vector and calculate the dot product to find her total interest in one year.

$$\begin{pmatrix} \_\_\_ & \_\_\_ & \_\_\_ \end{pmatrix} \begin{pmatrix} \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{pmatrix} =$$

4. N'Tanya's friend Shayna keeps her money in a different bank. Shayna's bank is offering 3% interest on regular savings account but they only pay 5% on certificates of deposit. Money market accounts earn the same amount at both banks. Use the dot product to find out the interest N'Tanya would earn in one year if her accounts were at Shayna's bank.

$$\begin{pmatrix} \_\_\_\_\_ & \_\_\_\_\_ & \_\_\_\_\_ \end{pmatrix} \begin{pmatrix} \_\_\_\_\_ \\ \_\_\_\_\_ \\ \_\_\_\_\_ \end{pmatrix} =$$

MATRIX MATH Days 3-4

Topic: Matrix Multiplication

Goals: 1. Know how the elements of a matrix product were calculated.

2. Know how to multiply the rows of one matrix times the columns of a 2nd matrix to obtain the product.

3. Know that matrix multiplication is not commutative.

Background for the teacher

On day 2, we saw how to compute the dot product, multiplying a row vector times a column vector. Today we will see that matrix multiplication is performed by multiplying each row from the matrix on the left times each column from the matrix on the right. The results obtained from multiplying the first row on the left times each column on the right is a set of numbers which becomes the first row of the answer:

$$\begin{pmatrix} 2 & 3 & 1 \\ * & * & * \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 & 4 \\ 3 & 1 & 5 & 2 \\ 2 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 13 & 7 & 16 & 15 \\ \_ & \_ & \_ & \_ \end{pmatrix}$$

Now the second row can be multiplied times each column to give a second row of 4 answers (product with first column is filled in…):

$$\begin{pmatrix} 2 & 3 & 1 \\ 5 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 & 4 \\ 3 & 1 & 5 & 2 \\ 2 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 13 & 7 & 16 & 15 \\ 14 & \_ & \_ & \_ \end{pmatrix}$$

The product matrix has one row of answers for each row in the left matrix and one column for each column in the right matrix. Above we multiplied a 2x3 and a 3x4 matrix together. The dimensions combine as follows:

2 x 3 times 3 x 4 = a 2 x 4 matrix product

What happened to the 3 in 2x3 and the 3 in 3x4? That is the number of things in each row on the left and the number of things in each column on the right. If they were not equal in length, we could not do the multiplication. When we do each dot product, the 3 pairs of products are summed together (so the two "3's" are collapsed into a single number in the product). A 5x7 and a 7x3 matrix can be multiplied (because of the two 7s) but a 5x6 and a 7x3 matrix cannot be multiplied (because 6 is not equal to 7).

Note that a 5x7 matrix can be multiplied times a 7x3 matrix but the reverse is not true: a 7x3 matrix cannot be multiplied times a 5x7 matrix (since 3 is not equal to 5). If the dimensions match (2x2 matrix times a 2x2 matrix), you can do the multiplication both ways, but the result will usually be different! (Matrix multiplication is not commutative.)

MATRIX MATH Day 3 Activity

Matrix Multiplication

Yesterday, we saw how to multiply a row times a column. If there is more than one column in the second matrix, we multiply the row times each of the columns. If there are 4 columns then there are _____ answers!

$$\begin{pmatrix} 2 & 5 & -3 \end{pmatrix}\begin{pmatrix} 1 & 3 & -1 & 2 \\ 2 & 4 & 0 & 1 \\ 2 & 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \end{pmatrix}$$

If we add another row to the matrix on the left, we get another row of answers. Find the dot product of $\begin{pmatrix} 4 & 0 & 1 \end{pmatrix}$ with each of the 4 columns and fill the answers into the blanks.

$$\begin{pmatrix} 2 & 5 & -3 \\ 4 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & -1 & 2 \\ 2 & 4 & 0 & 1 \\ 2 & 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 26 & -5 & 3 \\ \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \end{pmatrix}$$

Notice:

This first matrix has 2 rows, so the answer has 2 rows.

The 2nd matrix has 4 columns, so the answer has 4 columns.

Practice Matrix Multiplication: Pages of matrix multiplication problems can be found on the web.

MATRIX MATH Day 4 Activity

More Matrix Multiplication

Matrix multiplication is not commutative! (A*B is not the same as B*A) Finish the two products below to check this for yourself:

$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & \_ \\ \_ & \_ \end{pmatrix}$ and then change the order $\begin{pmatrix} 2 & 5 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & \_ \\ \_ & \_ \end{pmatrix}$

Understanding a Matrix Product

Look at the Matrix Product below. Pick an entry in the product matrix and verify that it is correct by finding the row and column that are multiplied. For example, the entry in the 2nd row, 3rd column is ____. (Circle the 2nd row of the left most matrix and the 3rd column of the matrix to its right. Compute the dot product to get the result in the 2nd row, 3rd column of the answer matrix.)

$$\begin{pmatrix} 2 & 0.5 & 10 \\ 3 & 0 & 5 \\ -1 & 1 & 6 \end{pmatrix}\begin{pmatrix} 6 & 1 & 3 & 0 \\ -8 & 2 & 2 & 6 \\ 0.2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 33 & 47 & 13 \\ 19 & 18 & 29 & 5 \\ -12.8 & 19 & 23 & 12 \end{pmatrix}$$

Now check a few more of the numbers in the answer. For example, the very first answer (1st row, 1st column) is $\begin{pmatrix} 2 & 0.5 & 10 \end{pmatrix}\begin{pmatrix} 6 \\ -8 \\ 0.2 \end{pmatrix} = 12 - 4 + 2 = 10$.

MATRIX MATH Day 5

Topic: Matrix Transformations in a Coordinate Plane

Goals:
1. Know how to place the x and y coordinates of a point into a column vector.
2. Multiply a 2x2 matrix times the vectors of all the corners (vertices) of a geometric shape and plot the resulting points to see the result of this transformation.
3. Know that the vectors of the corners can all be put into one matrix for faster calculation.

Background for the teacher

In this lesson we see how matrix multiplication can be used to "transform" a figure drawn on a coordinate plane. We first observe that the coordinates of a point in the xy plane form a vector with 2 elements. An example is the point (4,1). If we put the coordinates into a column vector (a 2x1 matrix) and multiply it by a 2x2 matrix on the left:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} \_\_\_\_\_ \\ \_\_\_\_\_ \end{pmatrix}$$

The result is another 2x1 vector $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ which we can plot in the plane as the point (5,1). Note that this simple operation gives us the idea that matrix multiplication can be used to move points in the plane (create an animation). The activity asks students to use the same matrix to transform all 4 corner points of a rectangle. When they plot the transformed points they will see that they get a new figure – in this case, a stretched version of the rectangle (= a parallelogram) which is moved a bit to the right. Visualize cartoons in which moving and stretching are key changes that convey the action.

MATRIX MATH Day 5 Activity

Matrix Transformations in a Coordinate Plane

Find the coordinates of the four points that define the rectangle in the grid on the left.

A = (______,______) TA = (______,______)

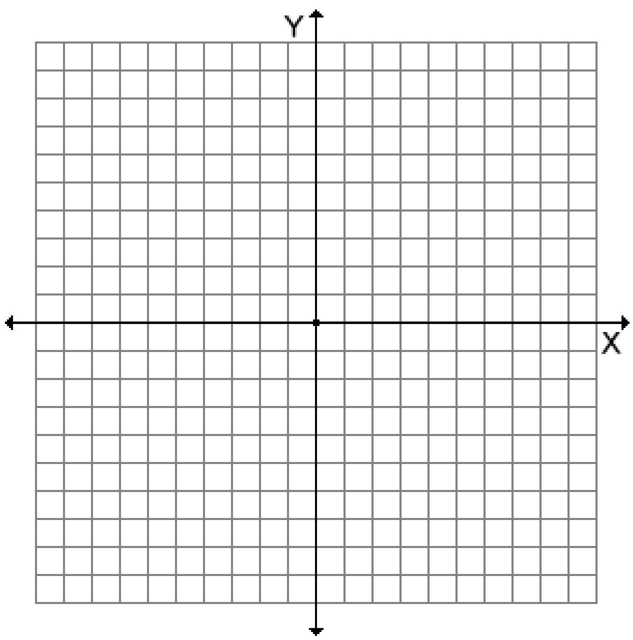
$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

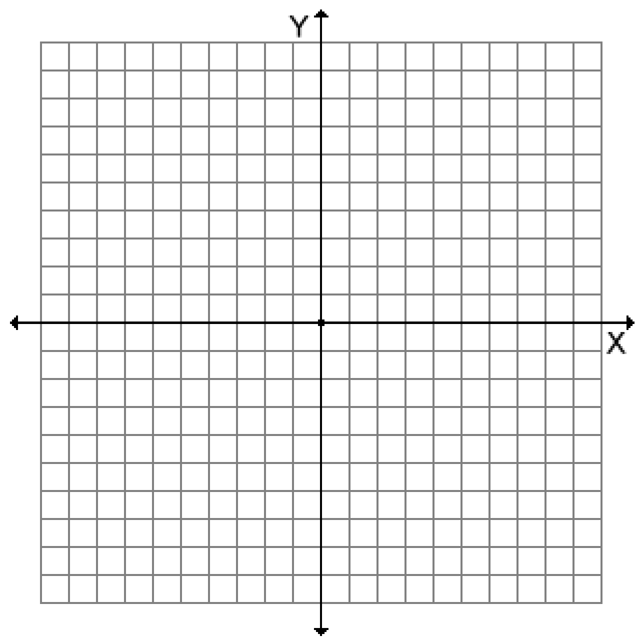
B = (______,______) ----------------------> TB = (______,______)

C = (______,______) TC = (______,______)

D = (______,______) TD = (______,______)

The original figure is shown here. Plot the transformed coordinate pairs here.





Show your work below.

Multiply the transformation matrix times the vectors of the coordinate pairs below to find the coordinates for the transformed object. How does it compare to the original?

A = (1,3)

Transformation of A: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \_\_\_\_\_ \\ \_\_\_\_\_ \end{pmatrix}$

B = (4,3)

Transformation of B: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} \_\_\_\_\_ \\ \_\_\_\_\_ \end{pmatrix}$

Day 6

Topic: Special Matrices

Goals: 1. Know how to find the identity matrix (the “1” of matrix multiplication).

2. Learn about the Zero matrix and other nilpotent matrices.

3. Learn how to identify diagonal, symmetric and skew symmetric matrices.

Background for the teacher

Some matrices have special roles in matrix algebra. We’ve seen that you can multiply two matrices, but the multiplication is not commutative. Clearly matrix multiplication is different from the multiplication of real numbers. Does everything we learned about ordinary multiplication have to be thrown out when we think about matrices? In a way, yes. We should not take anything for granted. But there are a lot of analogous features. Both the similarities AND the differences make matrix algebra an interesting and rich area of mathematics for applications and for study.

I start this lesson by multiplying (1 0 0) times a 3 x 4 matrix. Students quickly realize that you get the first row of the right hand matrix as your answer. Then we look at (0 1 0) and (0 0 1). Putting them all together:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & -1 & 2 & 7 \\ 0 & 5 & 3 & -4 \\ 8 & 0 & 6 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 2 & 7 \\ 0 & 5 & 3 & -4 \\ 8 & 0 & 6 & 1 \end{pmatrix}$$

This special matrix is called the IDENTITY MATRIX. It has exactly the same role as the 1 in real number multiplication. Make sure students realize that they can create an identity matrix of any (square) size. We denote the nxn identity matrix by I_n. $I_nA = A$ for any matrix A that has the right dimensions (n rows and any number of columns). And similarly, $AI_n = A$ for any matrix with n columns (do an example to see why).

The entries in an identity matrix are zero except on the main (top left to bottom right) diagonal. A matrix with this property is called a DIAGONAL MATRIX.

The ZERO MATRIX is easier. If every element of a matrix is zero, then all the dot products will be zero. So, $0A = 0$ (though we still need the dimensions to match or the equations does not make sense). A very interesting thing to note is that it is possible for the product of 2 matrices to be zero even if neither matrix is a zero matrix.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

If a power of a matrix (the matrix multiplied by itself 2 or more times) is equal to zero, we say that the matrix is NILPOTENT.

Matrix Math Day 6 Activity

Calculate the following product to illustrate that multiplication by the identity matrix times a square matrix of the right dimensions is commutatitve.

1. $\begin{pmatrix} 4 & 5 \\ 1 & 9 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 5 \\ 1 & 9 \end{pmatrix} = \begin{pmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{pmatrix}$

Now try it when the number on the diagonal is different from 1.

2. $\begin{pmatrix} 4 & 5 \\ 1 & 9 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 4 & 5 \\ 1 & 9 \end{pmatrix} = \begin{pmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{pmatrix}$

In this 3rd problem, the diagonal elements are different.

3. $\begin{pmatrix} 4 & 5 \\ 1 & 9 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 4 & 5 \\ 1 & 9 \end{pmatrix} = \begin{pmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{pmatrix}$

What is one thing we can learn from the 3 examples above?

Now multiply a matrix times itself:

4. $\begin{pmatrix} 4 & 5 \\ 1 & 9 \end{pmatrix}\begin{pmatrix} 4 & 5 \\ 1 & 9 \end{pmatrix} = \begin{pmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{pmatrix} \qquad \begin{pmatrix} 4 & 5 \\ 1 & 9 \end{pmatrix}\begin{pmatrix} 4 & 5 \\ 1 & 9 \end{pmatrix} = \begin{pmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{pmatrix}$

A matrix times itself is always commutative (A*A = A*A).

A SYMMETRIC MATRIX has the same rows as columns; SKEW SYMMETRIC has rows that are the negative of the columns:

SYMMETRIC: $\begin{pmatrix} 7 & 1 & 2 \\ 1 & -2 & 4 \\ 2 & 4 & 5 \end{pmatrix}$ SKEW-SYMMETRIC: $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 4 \\ -2 & -4 & 0 \end{pmatrix}$

Day 7

Topic: Rotation of the points (1,0) and (0,1)

Goals: 1. Rotate the point (1,0) 30 degrees and find its new coordinates.

2. Rotate the point (0,1) 30 degrees and find its new coordinates.

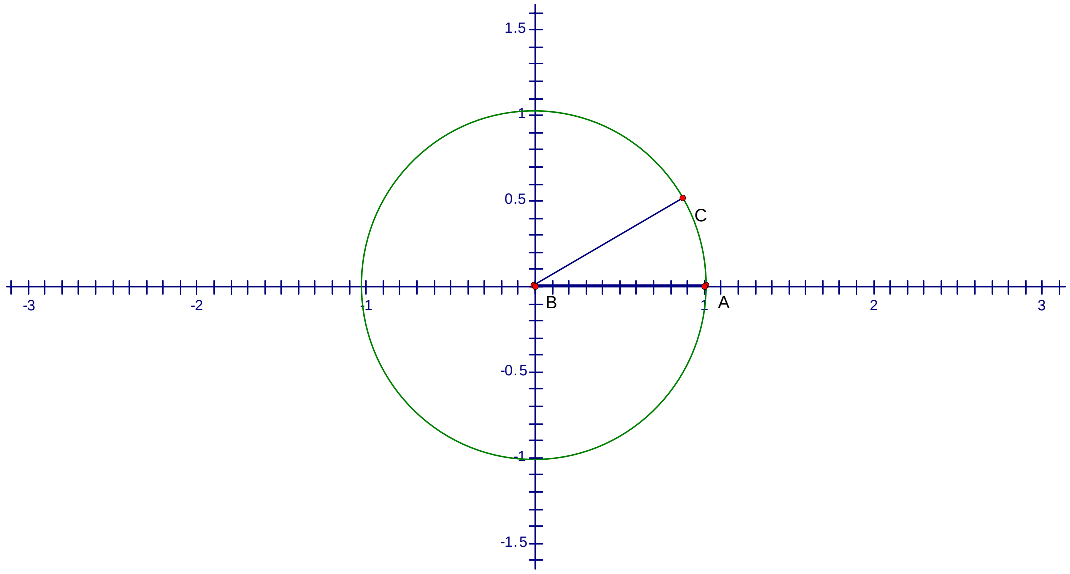
Background for the teacher

We saw that multiplying a 2x2 matrix times a column vector with the coordinates of a point in the xy plane produces a new point. In this lesson and the next one, we ask the question "If we want to move a point in the plane in a specific way, can we find the matrix that would do that?" We've chosen to explore this question for the example of a 30 degree rotation of a point around the origin. Rotations are an important component of animations (imagine a diver rotating and then extending, a hand turning a door knob, a gymnast doing a back flip, the Space Shuttle circling the earth). The first day, we just explore what it means to rotate a point. We plot two simple points (0,1) and (1,0) on a unit circle in the coordinate plane, and then find where those points would end up if they are rotated 30 degrees about the origin (that is, moved 30 degrees counter-clockwise around the unit circle). Finding a matrix to do this transformation is the next day's activity.

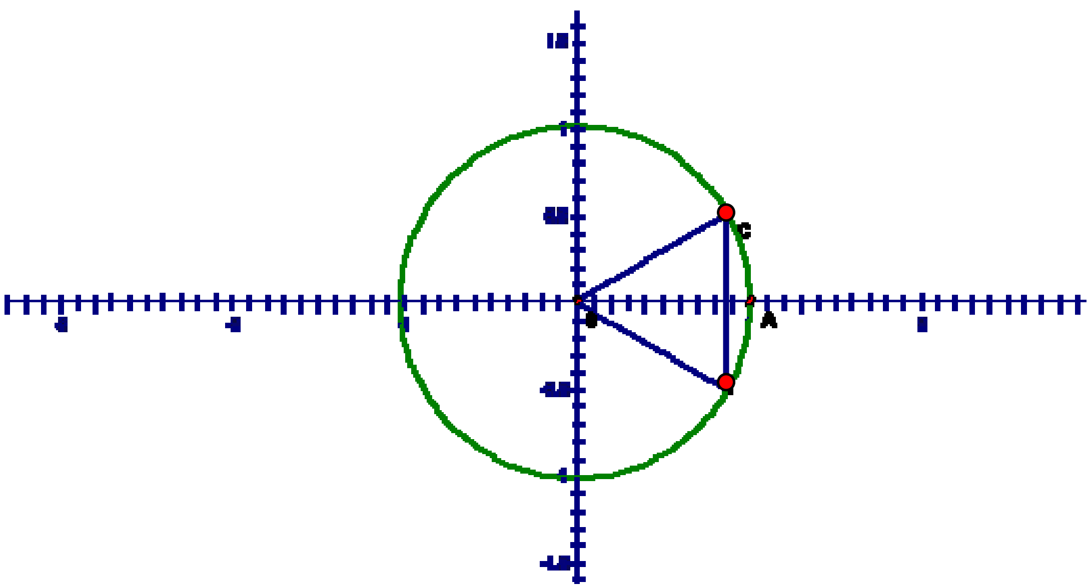
Note: An important observation is that the rotations of (1,0) and (0,1) are closely related problems, so the coordinates of the rotated points should also be related. It is useful to make an educated guess about how they would be related, and if they are not related in exactly the way we guessed, we can let the result inform our thinking about the geometry.

ROTATING THE POINT (1,0)

When we rotate the point (1,0) about the origin (the point (0,0)), it follows the path of the unit circle. If the point is rotated 30 degrees, then its new coordinates are the coordinates of the point on the unit circle where the radius makes an angle of 30 degrees with the x axis.



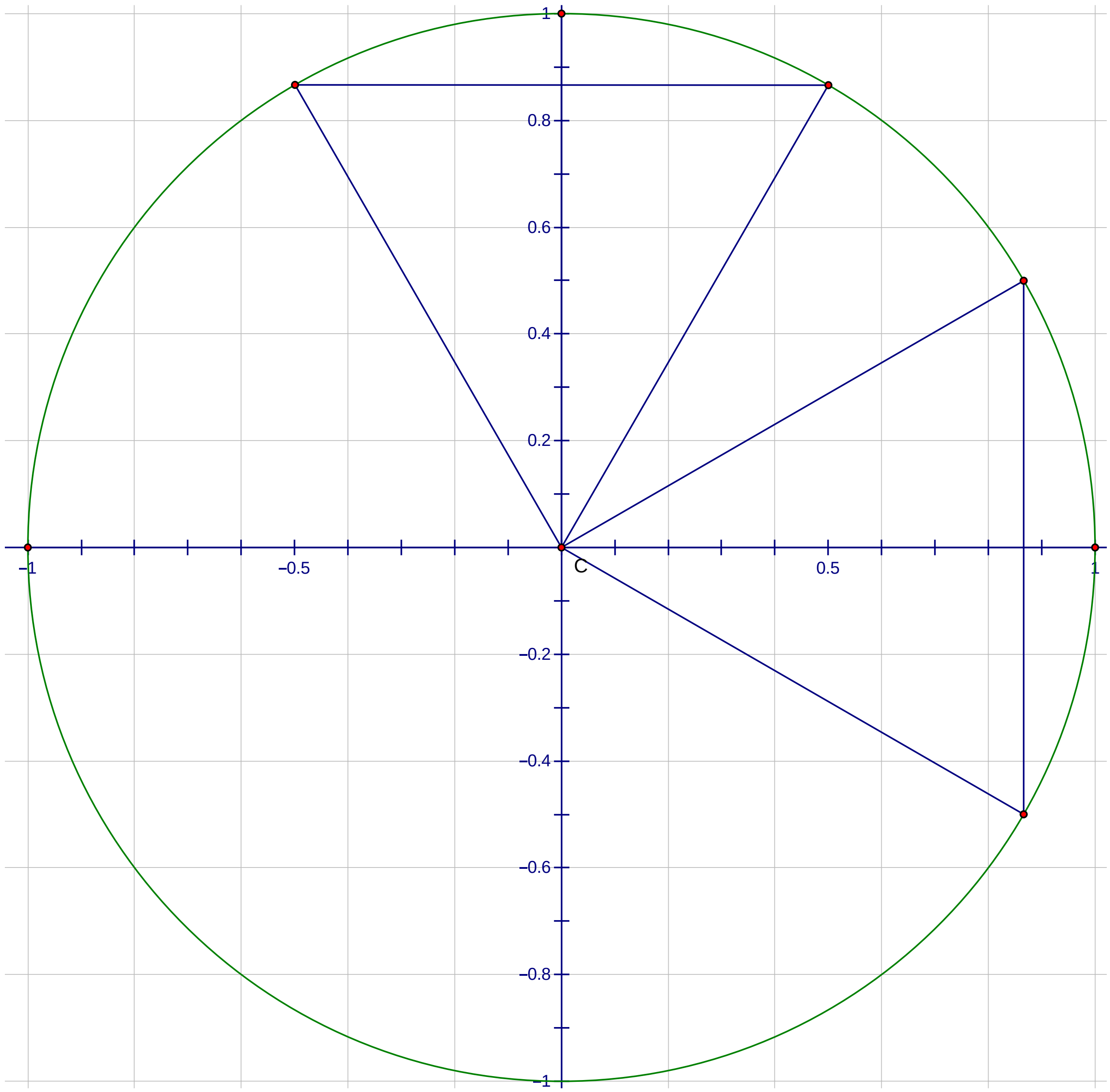
A simple way to find the new coordinates is to draw the same angle below the x axis and join the two points (forming a chord) to see where the x-coordinate falls on the axis. The y-coordinate is half the length of the chord.



Look closely and see if you can tell that the x-coordinate is about 0.87. In ancient India, the half-chord distance was given a name which eventually was translated into Latin as sinus. For the 30 degree angle above, check that the half-chord (sine of 30 degrees) is 0.5.

Rotating (1,0) and Rotating (0,1) are Complementary Problems

This plot can be used to visually identify the new coordinates (_______,_______) for a 30 degree rotation of (1,0). Next find the coordinates (_______,_______) for the 30 degree rotation of the point (0,1). The two triangles illustrate the complementary nature of the two problems.



Day 8

Topic: Matrix Transformation for the 30 degree Rotation

Goals: 1. Find a matrix transformation that moves (1,0) and (0,1) to the rotated positions.

2. Use this transformation to rotate a figure of your choice.

Background

Our goal is to find the transformation matrix that rotates the point A=(1,0) by 30 degrees around the origin (the point 0,0). Students found the points that represent a 30 degree rotation of (1,0) and of (0,1). The results were roughly (0.87, 0.5) and (-0.5, 0.87). As a group, work through the equations to find the matrix:

Step 1: Set up the equations

A = (1,0) 30 degree rotation of A: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.87 \\ 0.5 \end{pmatrix}$

B = (0,1) 30 degree rotation of B: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.5 \\ 0.87 \end{pmatrix}$

Step 2a: Solve these two equations. The equation for (1,0) yields a = 0.87 and c = 0.5. It gives no information about b and d. The equation for (0,1) yields b = -0.5 and d = 0.87. Note that we get values for all 4 unknowns and the two equations are compatible (no conflicting results).

Step 2b: Observe that these two equations could have been combined into a single matrix equation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.87 & -0.5 \\ 0.5 & 0.87 \end{pmatrix}$$

One advantage of this perspective is that we immediately recognize the 2nd matrix as the identity, so the equation is just $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0.87 & -0.5 \\ 0.5 & 0.87 \end{pmatrix}$.

Note: This activity could be carried out with any two distinct points in the plane that do not both lie on a straight line through the origin. However, the algebra would be more involved and the lesson would take longer. The independent student activity for this lesson emphasizes testing to see that the matrix determined by the rotation of these two points will rotate any point by 30 degrees. One of the goals of a linear algebra course would be to develop a theoretical foundation that would make it obvious that the solution for two points would uniquely identify the transformation. This short activity just gives a hint of this.

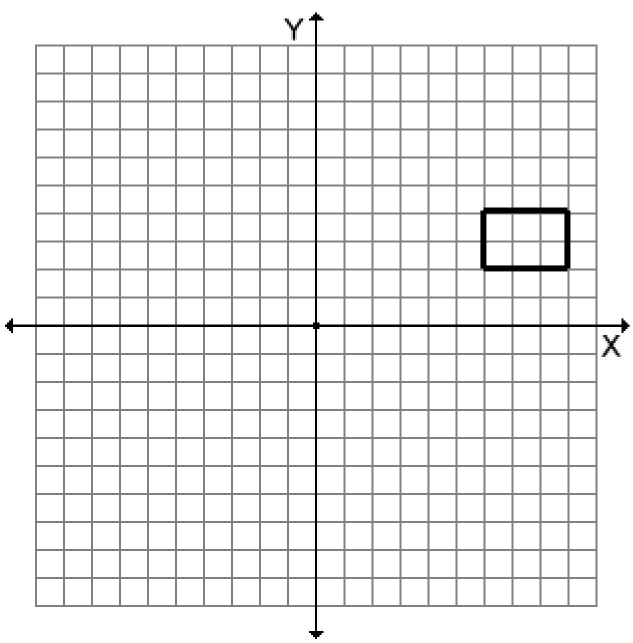
Matrix Math Day 8 Activity

The new transformation rotates (1,0) 30 degrees and it rotates (0,1) 30 degrees. What does it do when we transform another point in the plane? Will it rotate all points 30 degrees? To test this, plot a simple figure in the plane (a triangle or quadrilateral) with vertex points (x,y).

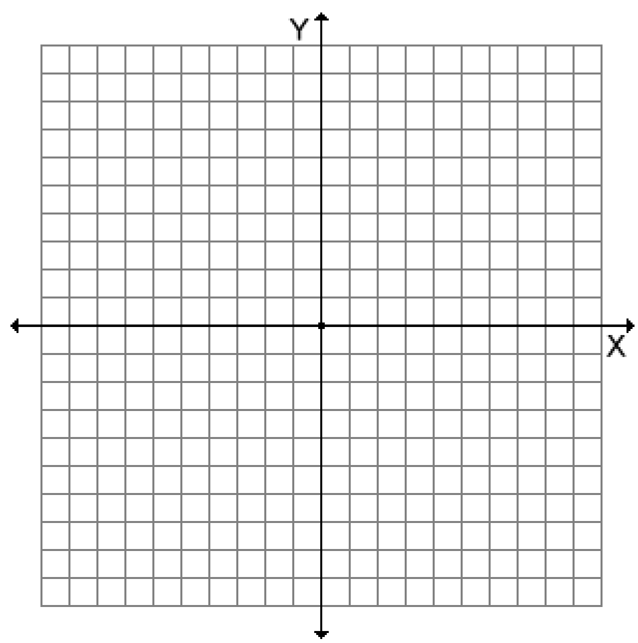
Exercise 1. Multiply the coordinate column of one of the vertices by the transformation matrix $\begin{pmatrix} 0.87 & -0.5 \\ 0.5 & 0.87 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ and plot the transformed point. Draw a line from the origin to (x,y) and another line from the origin to the transformed point. Use a protractor to measure the angle and use a ruler to measure the length of each line (or calculate the lengths using the Pythagorean thereom). Confirm that the point is rotated 30 degrees.

Exercise 2. Multiply the coordinate columns of all of the vertices by the transformation matrix and plot the new vertices. Visually confirm that the figure has been rotated about 30 degrees.

A figure to rotate is shown below.



Plot the transformed coordinate pairs here.



Day 9-10

Topic: Rotations of Any Degree – Sine and Cosine

Goals: 1. Find the coordinates (x=cosine; y=sine) for rotating (1,0) multiples of 15°.

2. Know how to form the transformation matrix for any rotation.

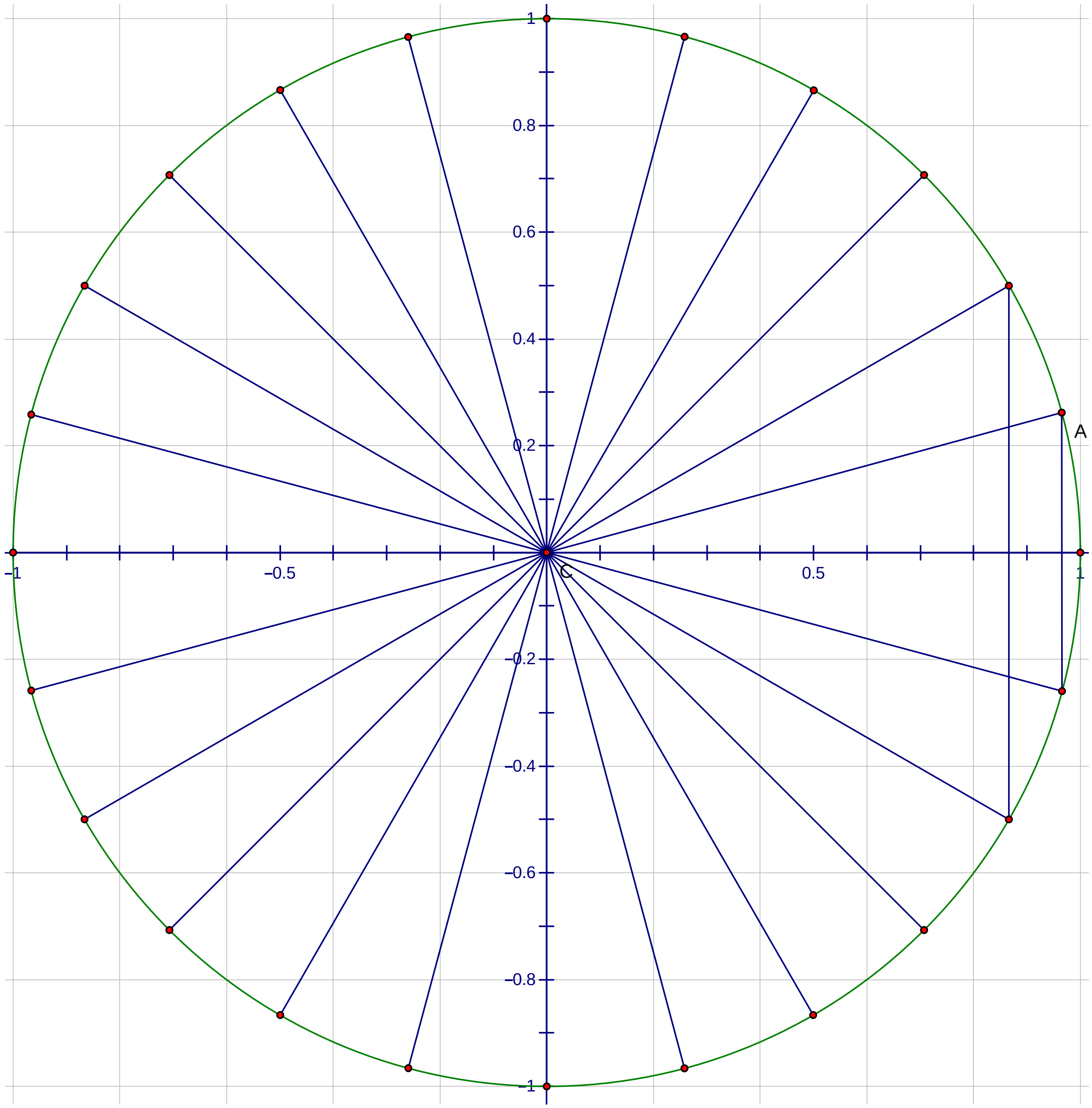
Background

The same process used to identify the matrix for a rotation of 30° can be used to get a rotation matrix for any size rotation. We use a unit circle with angles from 0° to 90° marked out in 15° increments to create a table of values for this purpose. The coordinates of the point at A° are cos A° and sin A° . The values can be determined visually and they can be confirmed using the sin and cos features on a scientific calculator – be sure the angle measurement unit is degrees.

The rotation matrix for d° is $\begin{pmatrix} \cos(A) & -\sin(A) \\ \sin(A) & \cos(A) \end{pmatrix}$.

ANGLE = A	Coordinates for the rotation of (1,0)	Coordinates for the rotation of (0,1)	Sin(A)	Cos(A)	Rotation Matrix
15 degrees	(______,______)	(______,______)			
30 degrees	(0.87, 0.5)	(-0.5, 0.87)	0.87	0.5	$\begin{pmatrix} 0.5 & -0.87 \\ 0.87 & 0.5 \end{pmatrix}$
45 degrees	(______,______)	(______,______)			
60 degrees	(______,______)	(______,______)			
75 degrees	(______,______)	(______,______)			

CIRCLE WITH ANGLES THAT ARE MULTIPLES OF 15 DEGREES



Days 11-12

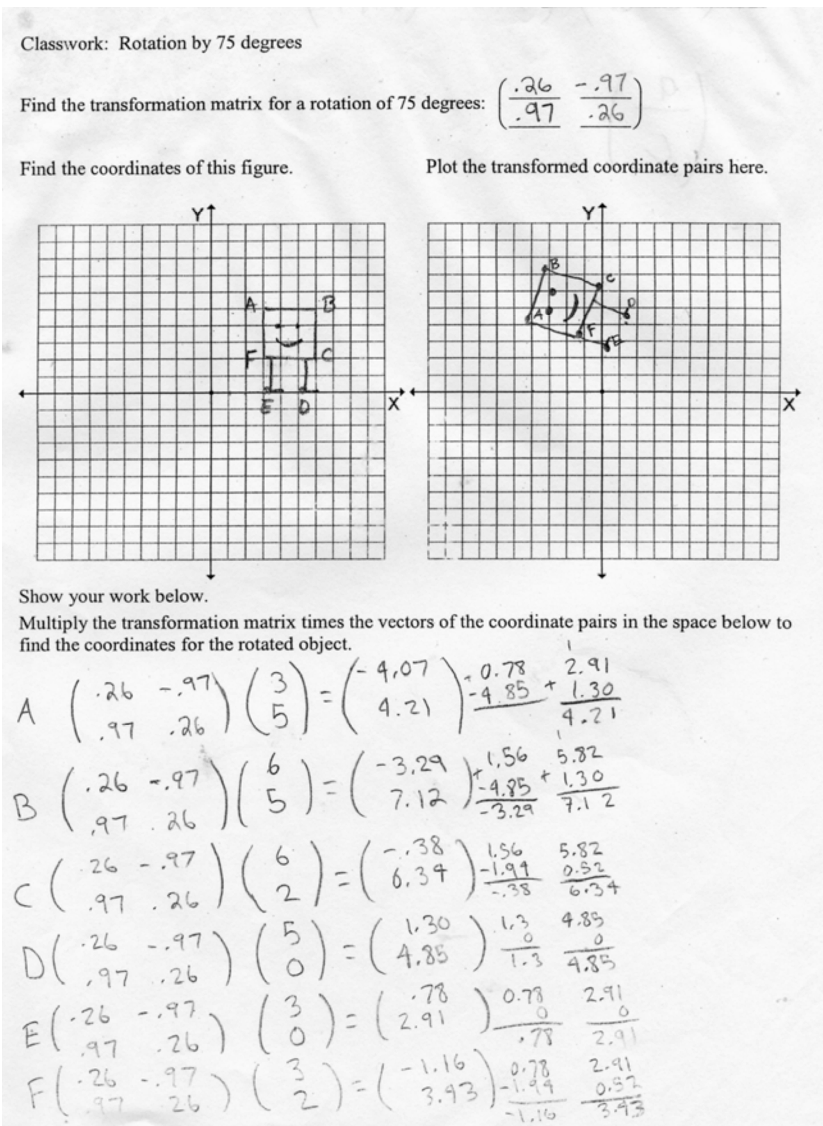
Topic: Rotate a figure by 75°.

Goals: 1. Find the coordinates of the figure.

2. Find the matrix for rotation by 75°.

3. Use this transformation matrix to rotate the figure in the xy plane.

These two days are devoted to practicing what was learned in the previous 3 days. Students rotate a simple figure given to them and then do the same for a simple figure of their own choosing. They should select the 75° matrix from the previous day's table, multiply the matrix times the column vector for each vertex of their figure, and plot the new transformed figure on a grid. A sample of work by an 8^{th} grade student is included below.


Classwork: Rotation by 75 degrees

Find the transformation matrix for a rotation of 75 degrees: $\begin{pmatrix} .26 & -.97 \\ .97 & .26 \end{pmatrix}$

Find the coordinates of this figure.

Plot the transformed coordinate pairs here.

Show your work below.

Multiply the transformation matrix times the vectors of the coordinate pairs in the space below to find the coordinates for the rotated object.

A $\begin{pmatrix} .26 & -.97 \\ .97 & .26 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -4.07 \\ 4.21 \end{pmatrix}$

B $\begin{pmatrix} .26 & -.97 \\ .97 & .26 \end{pmatrix}\begin{pmatrix} 6 \\ 5 \end{pmatrix} = \begin{pmatrix} -3.29 \\ 7.12 \end{pmatrix}$

C $\begin{pmatrix} .26 & -.97 \\ .97 & .26 \end{pmatrix}\begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} -.38 \\ 6.34 \end{pmatrix}$

D $\begin{pmatrix} .26 & -.97 \\ .97 & .26 \end{pmatrix}\begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.30 \\ 4.85 \end{pmatrix}$

E $\begin{pmatrix} .26 & -.97 \\ .97 & .26 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} .78 \\ 2.91 \end{pmatrix}$

F $\begin{pmatrix} .26 & -.97 \\ .97 & .26 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1.16 \\ 3.93 \end{pmatrix}$

Days 13-14

Topic: Review course material.

Goals: 1. Review the key concepts and skills from the course.

2. Answer practice questions.

Matrix Math Summary Activity Name______________________________

1. Give examples of 3x3 matrices: a) an identity matrix; b) a symmetric matrix; c) a diagonal matrix that is not an identity matrix.

2. Multiply the following row and column vector: $\begin{pmatrix} 3 & 5 & -2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ 2 \\ 7 \end{pmatrix}$

3. Multiply the matrix $\begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$

times the coordinates of the figure at right.

Then plot the transformed coordinates on the

same coordinate plane.

4. Explain how you find the elements of a rotation matrix. Find the transformation matrix for a rotation of 45 degrees in the xy plane.

5. Extra Credit

A = $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ B = $\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$

Find A*B and then B*A to show that matrix multiplication is not commutative.

Let $C = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Find C*C (which is C^2) and C*C*C (which is C^3) to show that C is nilpotent.