GEOMETRIC REPRESENTATIONS OF A 4X4 DETERMINANT

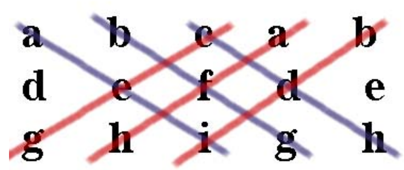
by
Dean Caffentzis

EXPANSION OF A THIRD ORDER DETERMINANT THROUGH EXPANSION BY DIAGONALS

Given the 3x3 matrix

a	b	c
d	e	f
g	h	i

copy columns 1 and 2 and add as columns 4 and 5 so



form the 6 3-tuple terms of the determinant by expansion:
aei + *bfg* + *cdh* (from left to right) and then *–bdi* – *afh* *–ceg* (from right to left)
A different approach is to evaluate the n! 3-tuple terms of the determinant which are found about their respective diagonal determinant terms. This is done using 2 maps of the 3x3 matrix,

a	b	c
d	e	f
g	h	i,

as follows…

Step 1: Find the determinant of the 3x3 through cofactor expansion; here we expand along the 1st column, so that for:

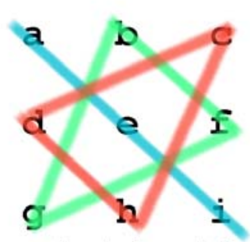
a b c
d e f
g h i (figure 1)

the determinant is, a(ei-fh) – d(bi-ch) + g(bf –ce) =
aei – afh – dbi + dch + gbf – gce

Step 2: We observe that for each term of the determinant:
aei, -afh, -dbi, dch, gbf, -gce, each i,j are represented uniquely for all a_{ij} in the set S of a 3x3 determinant, where aij may also be represented as:

11 12 13
21 22 23
31 32 33 (figure 2)

Step 3: The n! terms of 3-tuples are mapped (as an overlay) according to those terms that have similar patterns. For example, figure 1 above is overlayed with those determinant terms which are positive represented graphically as:



aei+dch+gbf

the negative are overlayed onto figure 1 as:


-afh-dbi-gce

We verify using figure 2 and according to the above graphs that for each term of the determinant, that no 2 elements belong to the same row or column.

COFACTOR EXPANSION OF A FOURTH ORDER DETERMINANT

a b c d
e f g h
i j k l
m n o p (figure 3)

Step 1: expanding along the 1st column yields,

a[(f(kp-lo) - j(gp-oh) + n(gl-kh)]
-e[(b(kp-lo) - j(cp-od) + n(cl-kd)]
+i[(b(gp-ho) - f(cp-od) + n(ch-gd)]
-m[(b(gl-hk) - f(cl-dk) + j(ch-gd)]

Step 2: when combining, the 24 terms, based on figure 3, are,

+afkp - aflo - ajgp + ajho + angl - ankh
-ebkp + eblo + ejcp - ejod - encl + enkd
+ibgp - ibho - ifcp + ifod + inch - ingd
-mbgl + mbhk + mfcl- mfdk- mjch + mjgd (figure 4)

Step 3: Start the overlay process of mapping the 24 determinant terms of figure 4 onto the matrix representation of figure 3.

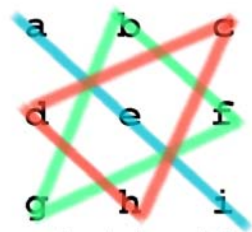
Step 4: By using the same pattern that were mapped in the 3x3, we match those terms to that of the determinant map of the 4x4

+afkp - aflo - ajgp + ajho + angl - ankh
-ebkp + eblo + ejcp - ejod - encl + enkd
+ibgp - ibho - ifcp + ifod + inch - ingd
-mbgl + mbhk + mfcl- mfdk- mjch + mjgd

using the patterns of the 3x3 mappings,

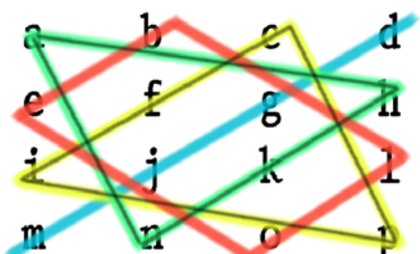
-afh-dbi-gce

and



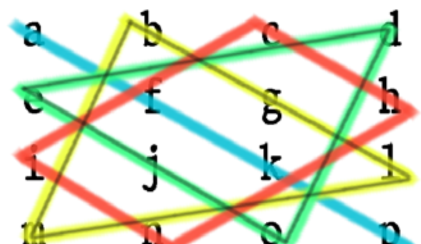
aei+dch+gbf

we see the 1st two maps of the 4x4, maps Ia and Ib as,



-ankh+ebol-ifcp+mjgd

and



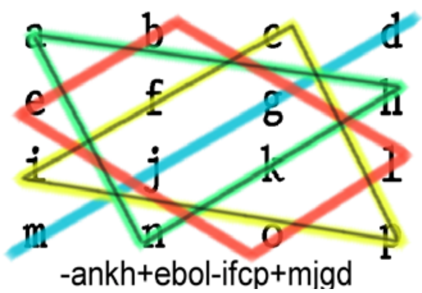
-dejo+chin-bglm+afkp

color coded here as:

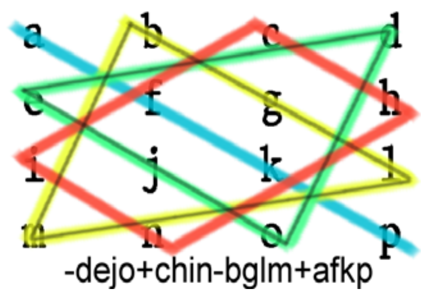
+afkp - aflo - ajgp + ajho + angl - ankh
-ebkp + eblo + ejcp - ejod - encl + enkd
+ibgp - ibho - ifcp + ifod + inch - ingd
-mbgl + mbhk + mfcl- mfdk- mjch + mjgd

When applying the pattern of the 3x3 to the 4x4 it is apparent that in the 3x3, both maps have 2 triangles and a diagonal and in the 4x4, maps Ia and Ib have 2 triangles, a diagonal and a rectangular parallelogram. Since however maps Ia and Ib of the 4x4 determinant only give us 8 out of the required 24 terms of the determinant we implement a new strategy so that all 24 terms can be represented.

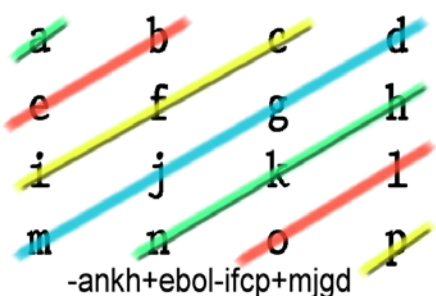
So instead of maps Ia and Ib patterned like this,



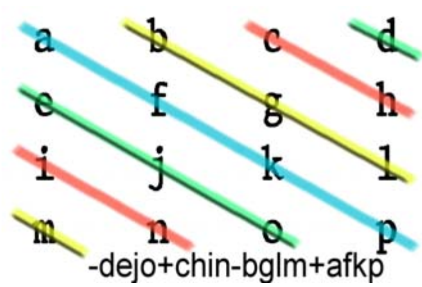
and



the elements of each term are connected with line segments through a diagonal (either the main diagonal, a lesser diagonal, or at a point).



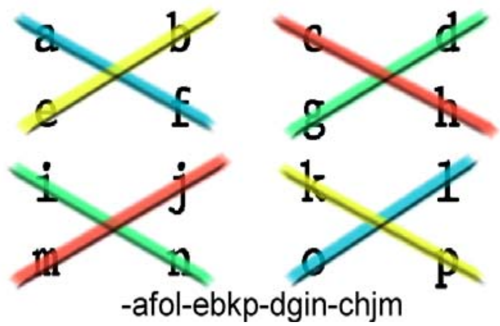
and



+afkp - aflo - ajgp + ajho + angl - ankh
-ebkp + eblo + ejcp - ejod - encl + enkd
+ibgp - ibho - ifcp + ifod + inch - ingd
-mbgl + mbhk + mfcl- mfdk- mjch + mjgd

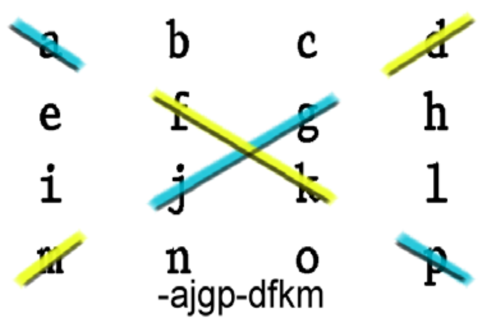
We continue our mapping strategy by overlaying the 4x4 represented by

a	*b*	*c*	*d*	
e	*f*	*g*	*h*	
i	*j*	*k*	*l*	
m	*n*	*o*	*p*	(figure 3) for map II as,



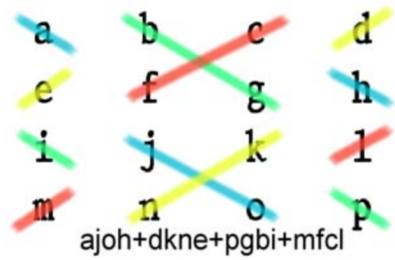
+afkp - aflo - ajgp + ajho + angl - ankh
-ebkp + eblo + ejcp - ejod - encl + enkd
+ibgp - ibho - ifcp + ifod + inch - ingd
-mbgl + mbhk + mfcl- mfdk- mjch + mjgd

and for map III as,



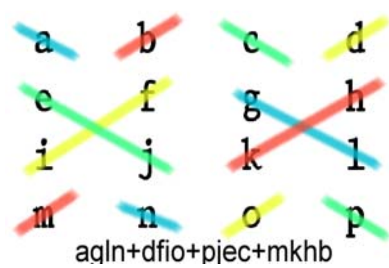
+afkp - aflo - ajgp + ajho + angl - ankh
-ebkp + eblo + ejcp - ejod - encl + enkd
+ibgp - ibho - ifcp + ifod + inch - ingd
-mbgl + mbhk + mfcl- mfdk- mjch + mjgd

The overlay of determinant terms for map IVa appears as,



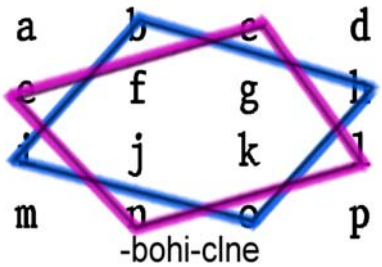
+afkp - aflo - ajgp + ajho + angl - ankh
-ebkp + eblo + ejcp - ejod - encl + enkd
+ibgp - ibho - ifcp + ifod + inch - ingd
-mbgl + mbhk + mfcl- mfdk- mjch + mjgd

while the overlay of determinant terms for map IVb look like,



+afkp - aflo - ajgp + ajho + angl - ankh
-ebkp + eblo + ejcp - ejod - encl + enkd
+ibgp - ibho - ifcp + ifod + inch - ingd
-mbgl + mbhk + mfcl- mfdk- mjch + mjgd

The final overlay pattern of determinant terms in map V is displayed graphically as 2 square parallelograms,

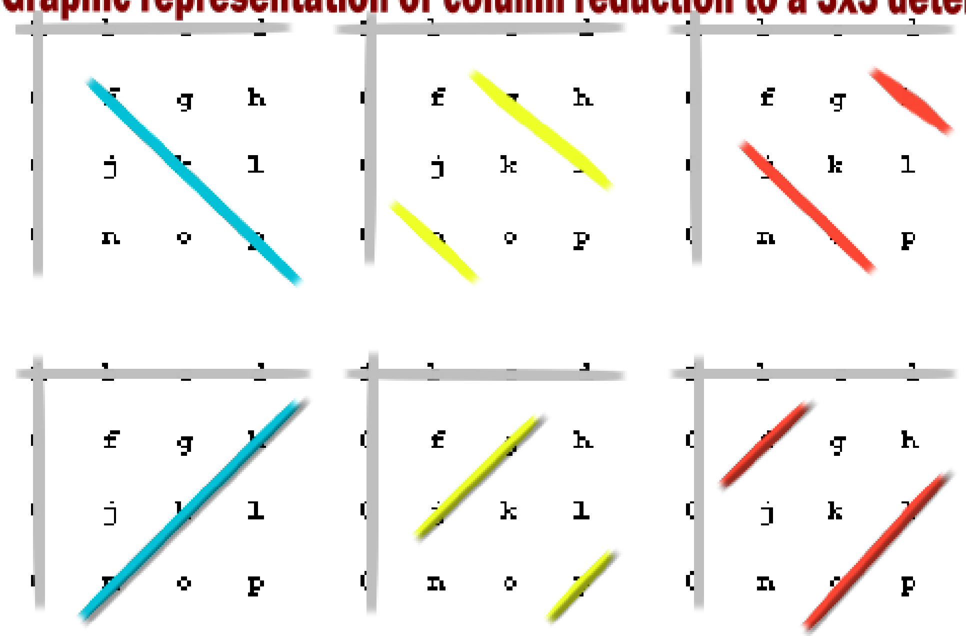


+afkp - aflo - ajgp + ajho + angl - ankh
-ebkp + eblo + ejcp - ejod - encl + enkd
+ibgp - ibho - ifcp + ifod + inch - ingd
-mbgl + mbhk + mfcl- mfdk- mjch + mjgd

We confirm that all the determinant terms of our 4x4 have been accounted for in maps Ia, Ib, II, III, IVa, IVb, and V by combining our color coded guide shown here as follows:

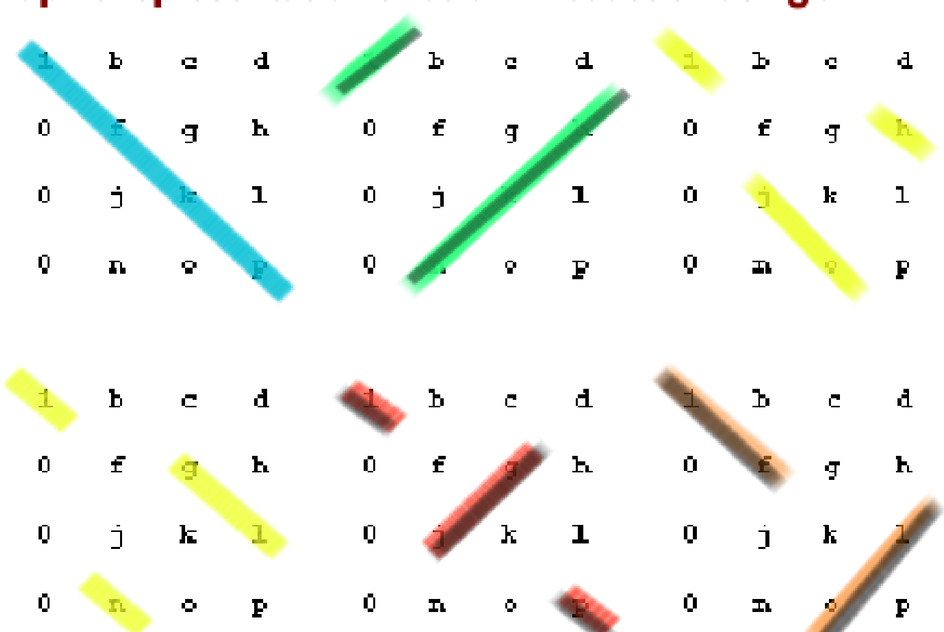
+afkp - aflo - ajgp + ajho + angl - ankh
-ebkp + eblo + ejcp - ejod - encl + enkd
+ibgp - ibho - ifcp + ifod + inch - ingd
-mbgl + mbhk + mfcl- mfdk- mjch + mjgd

Graphic representation of column reduction to a 3x3 determinant



fkp-nkh+joh+gln-jgp-fol

Graphic representation of column reduction using a 4x4 matrix



(1*fkp)-(1*nkh)+(1*joh)+(1*gln)-(1*jgp)-(1*fol)
=fkp-nkh+joh+gln-jgp-fol

$$/\det \lambda\mathbf{I}-\mathbf{A}/=$$

$$\begin{matrix} (\lambda-a) & b & c & d \\ e & (\lambda-f) & g & h \\ i & j & (\lambda-k) & l \\ m & n & o & (\lambda-p) \end{matrix}$$

Map Ia: mjgd-(λ-f)ci(λ-p)-h(λ-k)*
n(λ-a)+ebol

Map Ib: (λ-a)(λ-f)(λ-k)(λ-p)-bglm
-dejo+chin

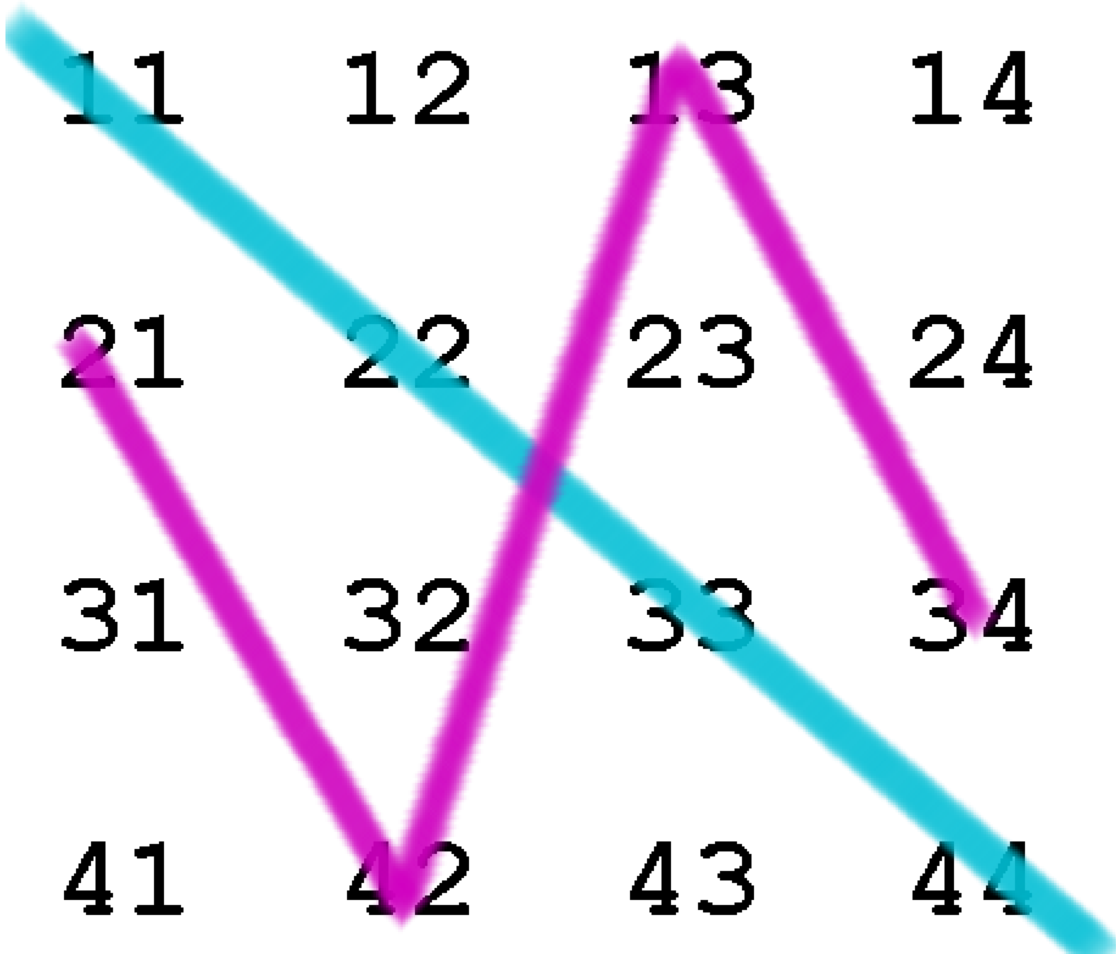
Map II: [(λ-a)(λ-f)ol]-[(λ-k)(λ-p)eb]
-chjm-dgin

Map III:[(λ-a)(λ-p)gj]-[(λ-f)(λ-k)md]

Map IVa:[(λ-a)joh]+[m(λ-f)cl]+
[(λ-p)gbi]+[d(λ-k)ne]

Map IVb:[(λ-a)gln]+[m(λ-k)hb]+
[(λ-p)jec]+[d(λ-f)io]

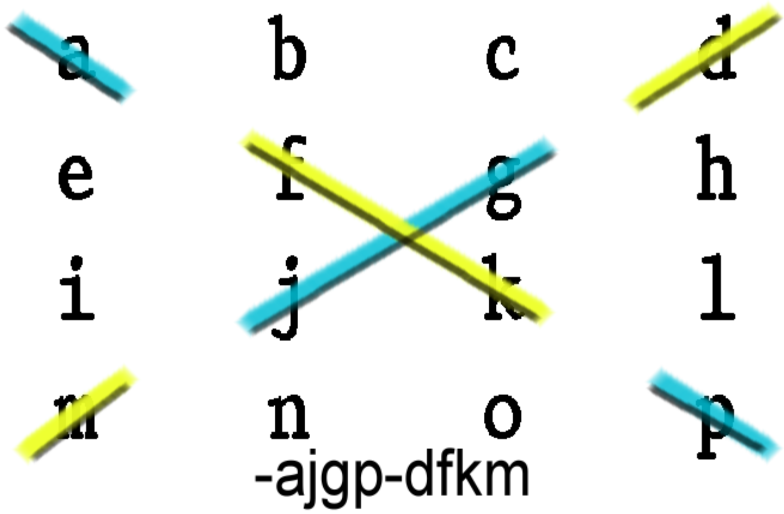
Map V: -clne–bohi

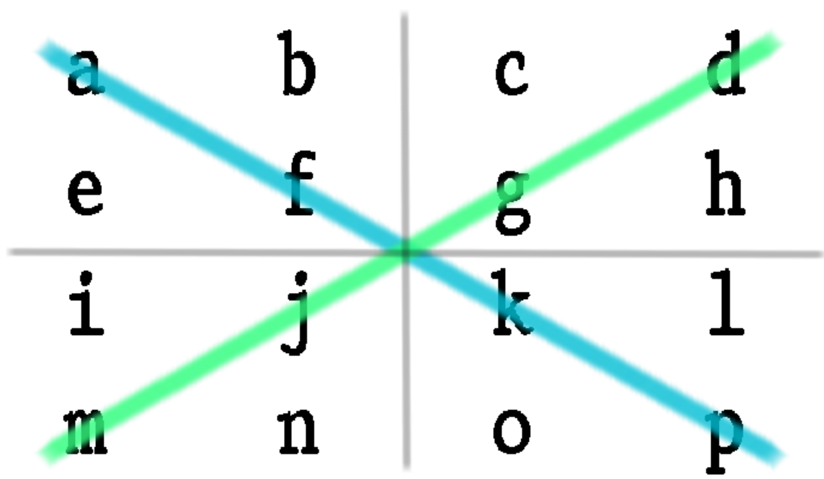


Composite trace with determinant term of map V

Composite trace with determinant term of map III

map III (complement map of map V)





The second complement of map V is an automorphism of the subset of elements in map III similar in construction to equations y = x and y = -x in R2 or as rotations of the xy axes.

If diag,{a,f,k,p=0}, then

0	**b**	**c**	**d**
e	**0**	**g**	**h**
i	**j**	**0**	**l**
m	**n**	**o**	**0**

Map Ia: dgjm + beol
Map Ib: -bglm –ejod +chin
Map II: -chjm -ingd
Map V: -cenl – bioh

Total nonzero terms det A = 9;
zeros = 15.

$$/\det \lambda I - A/ =$$

$$\begin{matrix} \lambda & b & c & \mathbf{0} \\ e & (\lambda-f) & g & h \\ \mathbf{i} & j & (\lambda-k) & l \\ \mathbf{0} & n & o & \lambda \end{matrix}$$

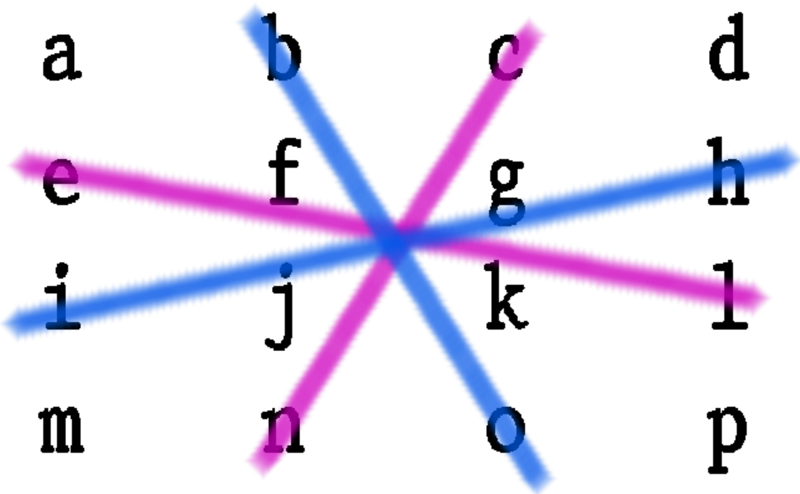
Map Ia: $-i(\lambda-f)c\lambda - \lambda nh(\lambda-k) + beol$
Map Ib: $\lambda^2[(\lambda-f)(\lambda-k)] + chin$
Map II: $-\lambda(\lambda-f)ol - \lambda(\lambda-k)eb$
Map III: $\lambda^2 jg$
Map IVa: $\lambda joh + \lambda gbi$
Map IVb: $\lambda gln + \lambda jec$
Map V: $-bioh - cenl$

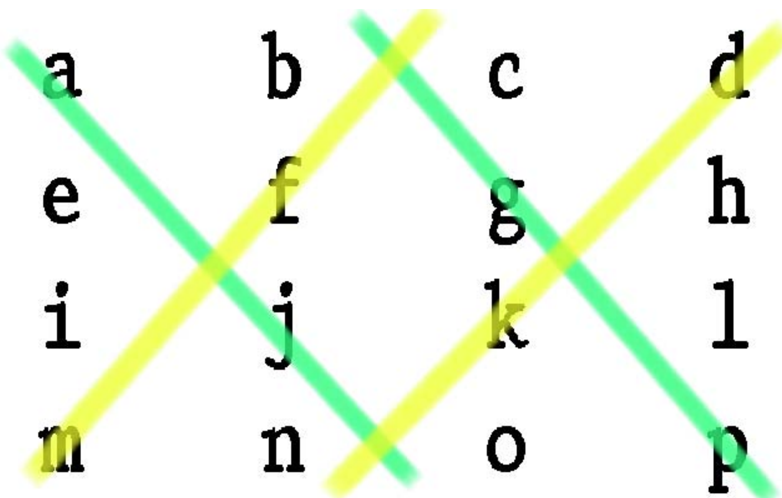
Characteristic equation of the zero corner group matrix:

$$= \lambda^2[(\lambda-f)(\lambda-k)+jg] + cn(hi-el) + bo(el-hi) - \lambda[(\lambda-f)ol+(\lambda-k)eb - joh - gbi - gln - iec] - \lambda[ci(\lambda-f)+nh(\lambda-k)]$$

$$= \lambda^2[(\lambda-f)(\lambda-k)+jg] + cn(hi-el) + bo(el-hi) - \lambda[((\lambda-f)(ol+ci)) - \lambda((\lambda-k)(eb+nh)) - joh - gbi - gln - iec]$$

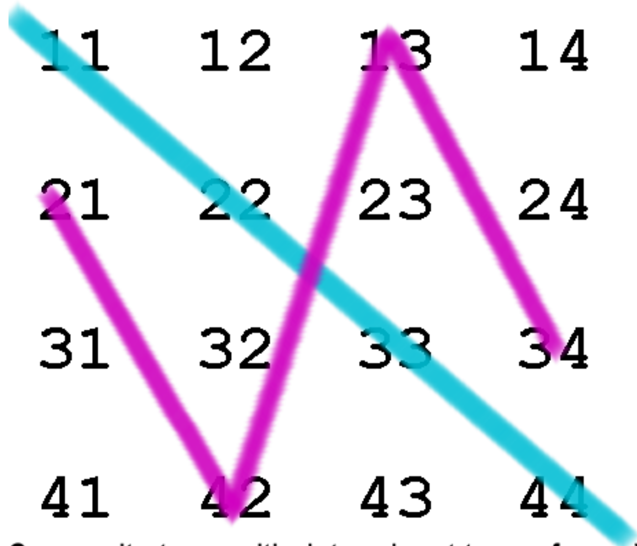
A second automorphism, here onto a subset of map V, where isomorphism F(β) is onto isomorphism F(α) where α,β are that part of the solution space of the characteristic equation of the “zero corner group” matrix. The geometry represented is intersecting lines at the origin.





It is easily verified that for that portion of the remainder of the solution space of the “zero corner group matrix”, the terms “oh” and ”eb” can be traced on both maps where the terms, ”ci” and ”nh” cannot be traced or mapped without separating out other elements.

APPROXIMATION METHODS AND PATTERNS OF CONVERGENCE



Composite trace with determinant term of map V

Composite trace with determinant term of map III

Maps V and III also viewed as a convergence onto the diagonal using composite maps of determinant terms. View the Darboux Integral (where Upper and Lower approaches to a curve or line) that parallel the sequence of processes that obtain an upper triangular or lower triangular square matrix in the finite case establishing those n x n forms as determinant functions.

Special Thanks to: My wife Carole and to my family and to Professor Judy Frasier, St. Petersburg College for meeting with me.

Biography: Dean Caffentzis, BA, Queens College, Flushing, NY 1984, Florida Insurance Licenses; 220,216.